TOPOLOGY OF ALGEBRAIC VARIETIES

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Abstract. Let Y be a normal crossing divisor in the smooth projective algebraic variety X (defined over $\mathbb{C}$) and let U be a tubular neighbourhood of Y in X. We construct homological cycles generating $H_*(A,B)$, where (A,B) is one of the following pairs $(Y,\emptyset)$, (X,Y), $(X,X-Y)$, $(X-Y,\emptyset)$ and $(\partial U,\emptyset)$. The construction is compatible with the weights in $H_*(A,B,\mathbb{Q})$ of Deligne's mixed Hodge structure.

I. Introduction

In his fundamental article [L], Leray gives a cohomological version of the theory of residues along a smooth hypersurface in an analytic manifold. The original motivation of the present article was the generalization of the theory of residues, but now along a normal crossing divisior (*NCD*) Y of a smooth projective algebraic variety X (defined over $\mathbb{C}$). Leray's construction of the inverse image of a cycle on Y leads to a homological cycle of the boundary of a tubular neihgbourhood of Y. In our case, in order to obtain a cycle as inverse image, we need to impose specific condition on the cycle on Y. We obtain such conditions by duality from Deligne's mixed Hodge structure (*MHS*) on the cohomology of algebraic varieties [D].

Once this fact has been well understood, we realised that we needed to develop a complete homological theory of some complexes (compatible with the natural mixed Hodge structures), and to describe the topology/homology of the pairs $(Y,\varnothing)$, (X,Y), $(X,X-Y)$, $(X-Y,\varnothing)$ and $(\partial U,\varnothing)$, where U is a tubular neihgbourhood of Y in X. These results are presented in this paper. The generalization of the residue theorems will be published elsewhere.

The reader may have recognized that the reduction to the *NCD* case uses Hironaka's desingularization theorem [H] in order to carry our results to smooth open algebraic varieties. In the cohomology theory, the main ingredient is Deligne's mixed Hodge structure. By duality, one can define the weight filtration in homology with $\mathbb{Q}$-coefficients. Our ultimate result is the construction of representative homological cycles according to their weight. Such constructions have been used occasionally in examples or some specific situations. Our objective is a systematic treatment giving a general account of the topology of algebraic varieties.

¿From the point of view of the present article, the key argument is the degeneration at rank two of the weight spectral sequence defined by the mixed Hodge complex, valid only for algebraic varieties. This in turn leads us to consider homological complexes

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with good intersection theory and for this it has been necessary to use the simplicial theory provided by the triangulations of varieties instead of singular chains.

The paper is organized as follows: in the introduction we recall some known facts about the topology of a regular neighbourhood of a *NCD*. We refer to the articles by Clemens [C] and A'Campo [A]. In the second part of this introduction, we already present the main steps of our construction.

In the second section we consider the complex of "dimensionally transverse" sub-analytic chains. This is in fact the complex of the sub-analytic "intersection chains" associated with the natural stratification of Y and zero perversity (in the sense of Goresky–MacPherson [GM]). Moreover, we define and analyze different chain morphisms whose role is absolutely fundamental in the construction of cycles.

Using the usual and the "dimensionally transverse" chains, in section 3 we define several homological double complexes $A_{**}(A,B)$ corresponding to the above pairs of spaces (A,B). They carry natural weight filtrations, giving rise to the corresponding spectral sequences. For example, in the case of $(X-Y,\varnothing)$ it is possible to show that our spectral sequence is dual to the weight spectral sequence of the logarithmic complex via residues. In this section we prove for all of the considered pairs the degeneration property at rank 2 of the weight spectral sequence. In section 4 we review some of the algebraic properties of a double complex whose spectral sequence degenerates at rank 2.

Using the result of the previous two sections, in section 5 we construct cycles compatible with the weight filtration. Here an additional important ingredient is the construction of chain morphism $Tot_*(A_{**}(A,B)) \to C_*(A,B)$ for any pair (A,B). This is based on the results of section 3. (Here Tot_* stays for the total complex, and $C_*(A,B)$ is the complex of sub-analytic chains of (A,B).)

The case of $(\partial U,\varnothing)$ is the most involved, so we separated it in section 6.

I.1. Now, we start a more detailed presentation.

I.1.1. **The stratification of Y.** Let X be a smooth projective algebraic variety containing a normal crossing divisor Y. Let $(Y_i)_{i\in I}$ be the decomposition of Y into irreducible components. Here the index set I is a totally ordered set.

The divisor Y has a natural stratification by subspaces Y^j consisting of "points of multiplicity $\geq j$ in Y", i.e.:

$$Y^j = \bigcup_{\alpha_1<\cdots<\alpha_j} Y_{\alpha_1}\cap\cdots\cap Y_{\alpha_j}, \quad \text{with normalization} \quad \tilde{Y}^j = \coprod_{\alpha_1<\cdots<\alpha_j} Y_{\alpha_1}\cap\cdots\cap Y_{\alpha_j}.$$

Sometimes we will use the notation $Y_{\alpha_1,\dots,\alpha_j} = Y_{\alpha_1}\cap\cdots\cap Y_{\alpha_j}$ as well. It is convenient to write $\tilde{Y}^0 = Y^0 = X$.

I.1.2. **The tubular neighbourhood of Y.** For any variety X and divisor Y, X admits a triangulation compatible with Y, hence there exists a regular (tubular) neighbourhood in X which is a deformation retract of Y (see e.g. [R-S]). In the case of a *NCD*, the tubular neighbourhood U of Y and a projection from U to Y can have even some additional properties, see [C].

A' Campo [A] describes such a neighbourhood as follows. The real (oriented) blow-up $\Pi_i : Z_i \to X$ with center Y_i provides a differentiable manifold Z_i with boundary and a projection Π_i on X, inducing a diffeomorphism outside Y_i. Moreover, $\Pi_i^{-1}(Y_i)$ is an S^1-bundle associated to the oriented normal bundle $N_{Y_i/X}$. We can identify the fibers at a point y in Y_i with the set of real oriented normal directions to Y_i. For example, if Y_i is a point in $\mathbb{C}$, then Z_i is a half-cylinder with boundary S^1 over the point. In general, the boundary of Z_i, equal to $\Pi_i^{-1}(Y_i)$, is diffeomorphic to the boundary of the complement of an open tubular neighbourhood of Y_i in X.

Next we consider the fibered product of the projections $(\Pi_i)_{i\in I}$ over X. Thus we obtain $\Pi : Z \to X$, where Z is a manifold with corners whose boundary ∂Z equal to $\Pi^{-1}(Y)$.

Here Z is homeomorphic to the complement of an open tubular neighbourhood U of Y in X, hence ∂Z is homeomorphic also to the boundary of the closure of U. In fact U is homeomorphic to the mapping cylinder of $\Pi|\partial Z : \partial Z \to Y$.

In order to see this, let's fix a homeomorphism $\phi_\alpha : \partial Z \times [0, \alpha] \to Z_\alpha$, where Z_α is some collar, and $\phi_\alpha|\partial Z \times \{0\}$ is the inclusion $\partial Z \hookrightarrow Z_\alpha$. There is a natural projection $p_\alpha : Z_\alpha \to \partial Z$ given by $pr_1 \circ \phi_\alpha^{-1}$, where pr_1 is the projection on the first factor. (This retract can be extended to a strong deformation retract of the inclusion $\partial Z \hookrightarrow Z_\alpha$.) Using this, one can define tubular neighbourhoods of Y in X as follows.

For any $\epsilon \in (0, \alpha]$, define $U_\epsilon := \Pi(\phi_\alpha(\partial Z \times [0, \epsilon]))$. Its boundary ∂U_ϵ is clearly $\Pi(\phi_\alpha(\partial Z \times \{\epsilon\}))$. Moreover, there is a unique map $q_\epsilon : U_\epsilon \to Y$ such that $q_\epsilon \circ \Pi = \Pi \circ p_\epsilon$. From the definitions follows that for two neighbourhoods $U_{\epsilon_1} \overset{i}{\hookrightarrow} U_{\epsilon_2}$, for $0 < \epsilon_1 < \epsilon_2$, one has $q_{\epsilon_2} \circ i = q_{\epsilon_1}$. This shows, that one can construct a uniform family of tubular neighbourhoods U_ϵ of Y in X, all of them homeomorphic to the mapping cylinder of $\Pi|\partial Z$, with all the natural compatibilities. This construction is rather satisfactory in the discussion of topological invariants.

But, in fact, Z has a natural semi-analytic structure (see II.3.1), and the above homeomorphism ϕ_α can be chosen as a sub-analytic homeomorphism as well (in fact, even as a piecewise analytic isomorphism with respect to the natural stratification). For the proof of this last statement, see [P]. Therefore, all the homeomorphisms, discussed in the previous paragraphs, can be considered as sub-analytic homeomorphisms.

I.1.3. **The construction of the double complexes $A_{**}(A, B)$.** For any $p \geq 0$, we denote the complex of the usual (respectively the "dimensionally transverse") sub-analytic chains (cf. II.1) defined on $\tilde{Y}^p$ by $(C_*(\tilde{Y}^p), \partial)$ (respectively by $(C_*^{\pitchfork}(\tilde{Y}^p), \partial)$). One has the following chain morphisms:

$$\begin{array}{ccccccccc}
\cdots & C^{\pitchfork}_{*-4}(\tilde{Y}^2) & \overset{\cap}{\longleftarrow} & C^{\pitchfork}_{*-2}(\tilde{Y}^1) & \overset{\cap}{\longleftarrow} & C^{\pitchfork}_{*}(\tilde{Y}^0) & & & \\
 & & & & & \Big\downarrow j_1 & & & \\
 & & & & & C_*(\tilde{Y}^0) & \overset{i}{\longleftarrow} & C_*(\tilde{Y}^1) & \overset{i}{\longleftarrow} \; C_*(\tilde{Y}^2) \; \cdots
\end{array}$$

Then define (modulo some shift of indexes): $A_{**}(X, X - Y) = (C_*^{\pitchfork}(\tilde{Y}^p), \partial, \cap)_{p\geq 1}$, $A_{**}(X-Y) = (C_*^{\pitchfork}(\tilde{Y}^p), \partial, \cap)_{p\geq 0}$, $A_{**}(Y) = (C_*(\tilde{Y}^p), \partial, i)_{p\geq 1}$, $A_{**}(X, Y) = (C_*(\tilde{Y}^p), \partial, i)_{p\geq 0}$,

and finally $A_{**}(\partial U)$ by the cone of the morphism j_1 in the above diagram where we replace $\tilde{Y}^0$ by U. The weight filtration W_* of a double complex A_{**} is defined by $W_s := \oplus_{p\le s} A_{pq}$.

I.1.4. **The construction of the cycles.** First, using the geometry of the pair (X,Y) (e.g. the projection Π and intersections corresponding to the stratification), for any pair (A,B) we construct a quasi-isomorphism $m_{A,B} : Tot_*(A_{**}(A,B)) \to C_*(A,B)$. Then any representative c_{st} of an element $[c_{st}] \in ker(d^1|E^1_{st})$ is completed to a chain $c^\infty_{st} = c_{st} + c_{s-1,t+1} + \cdots \in Z^\infty_{st}$ with $D(c^\infty_{st}) = 0$, where D is the differential of the associated total complex $Tot_*(A_{**})$ (and $c_{pq} \in A_{pq}$). Therefore, $m_{A,B}$ associates with any $[c_{st}]$ a closed cycle $m_{A,B}(c^\infty_{s,t})$ of dimension $k = s+t$.

For the various pairs (A,B) considered above, we have the following result.

I.1.5. **Theorem.**

a) The first term (E^1_{st}, d^1) of the weight spectral sequence can be explicitly determined from the homology of the spaces $\tilde{Y}^p$ and from the various normal bundles of the components of $\tilde{Y}^p$ in $\tilde{Y}^{p-1}$.

b) $E^r_{st} \Longrightarrow H_{s+t}(A,B,\mathbb{Z})$, and induces a weight filtration on the integer homology. Moreover $E^\infty_{st} \otimes \mathbb{Q} = Gr^W_{-t} H_{s+t}(A,B,\mathbb{Q})$ (the last considered in Deligne's MHS).

c) $E^*_{**} \otimes \mathbb{Q}$ degenerates at rank 2.

d) The above construction provides all the cycles of the pair (A,B) (modulo boundary) according to their weights. More precisely, the homology class $[m_{A,B}(c^\infty_{s,t})]$ of dimension $k = s+t$ is well-defined modulo $W_{-t-1}H_k(A,B,\mathbb{Q})$, and these type of classes generate $W_{-t}H_k(A,B,\mathbb{Q})$.

Here we refer the reader to theorem III.0 and propositions V.2.2, VI.2.3 and VI.2.4 for precise statements.

I.1.6. **Example.** Suppose $Y = Y_1 \cup Y_2$ has two components with smooth intersection $Y_{1,2} = Y_1 \cap Y_2$. Two homology classes $[a_i] \in H_k(Y_i)$, $i = 1,2$, satisfying $[a_1] \cap [Y_{1,2}] = [a_2] \cap [Y_{1,2}]$ in $H_{k-2}(Y_{1,2})$ give rise in ∂U to a $k+1$ dimensional cycle (generating a homology class of weight $-k-2$). Indeed, first we can assume that the representative a_i is transversal to $Y_{1,2}$ in Y_i. Due to the condition about $[a_i] \cap [Y_{1,2}]$, there exists a chain $a_{1,2}$ in $Y_{1,2}$ such that $\partial a_{1,2} = a_2 \cap Y_{1,2} - a_1 \cap Y_{1,2}$. Since dim $a_i = k$, we have:

$$\dim a_i \cap Y_{1,2} = k-2,\ \dim a_{1,2} = k-1,\ \dim \Pi^{-1} a_i = k+1,\ \dim \Pi^{-1} a_{1,2} = k+1.$$

Moreover, by the very construction, $\Pi^{-1}(a_1 + a_2 + a_{1,2})$ has no boundary. It is the wanted closed cycle in ∂U.

Now, consider the case when Y has three irreducible components Y_i $(i = 1,2,3)$ with $Y_{1,2,3} \neq \varnothing$. Similarly as above, we want to lift some closed cycles a_i into ∂U. The first obstruction is $[a_i \cap Y_{i,j}] = [a_j \cap Y_{i,j}]$ in the homology of $Y_{i,j}$ (for any pair $i \neq j$). Using this, we create the new chains $a_{i,j}$ in $Y_{i,j}$ as above. Now, if we want to lift these new chains and glue them together, we face the second obstruction provided by the triple intersection $Y_{1,2,3}$. The main point is that *there exists a good choice of $a_{i,j}$ such that this new obstruction is trivial*; its triviality is equivalent with the vanishing of

the second differential d_2 of the spectral sequence. Basically, this is the main message of the present paper.

II. Topological Preliminaries

II.1. **Geometric chains.**

II.1.1. **Preliminary remarks.** In some cases it is not absolutely evident how can we dualize a result established in cohomology. For example, if we want to compute the homology of Y, then Deligne spectral sequence in cohomology has a very natural analog in homology. On the other hand, if we want to find the homological analogue of the cohomological theory of $X-Y$, then we have to realize that there is no obvious homological candidate. Actually, the E_1-term of the spectral sequence associated with the log complex $(\Omega^*_X(\log Y), W)$ can be easily dualized; but we want (and need to) dualize the whole spectral sequence, in particular we have to construct the E^0 term on X. This is crucial in the construction of cycles in $X-Y$ as well.

Since the *Gysin differential*, in the cohomological E_1-term of $X-Y$, dualizes to the intersection of cycles, we need to work with *chains with good intersection properties* with respect to the stratification defined by Y. Similarly as in the case of the intersection homology groups, we have several options to define our chain complex. In the original definition of the intersection homology groups, Goresky and MacPherson used geometric chains with some restrictions provided by the perversities [GM]. On the other hand, H.King recovered these groups using singular chains [K]. We will follow here the first option (in fact, we will use sub-analytic geometric chains). In order to have a good intersection theory, we need some kind of transversality property. Here again we have several possibilities. Our choice asks only a "dimensional transversality" of the chains. This has the big advantage that the complex of these chains coincides with the complex of zero perversity chains of Goresky–MacPherson; but has the disadvantage, that in the intersections of the cycles we have to handle an intersection multiplicity problem.

Finally, our choice for *sub-analytic* chains is motivated by the fact that these chains are stable with respect to the real blowing up along Y (in contrast with the P.L. chains).

II.1.2. **Good class of subsets and chains.** Before we start the precise definition of our geometric chains, let us review briefly the general theory. We will follow the presentation from [G], pages 146-155.

Fix a manifold M. The group of *geometric chains* is defined in three steps. First, one defines *a good class* $\mathcal{C}$ *of subsets of* M. This should satisfy the following properties:

(1) if a subset S of M is in $\mathcal{C}$, then M has a Whitney stratification such that S is a union of strata, and each stratum is in $\mathcal{C}$;

(2) the class $\mathcal{C}$ is closed under unions, intersections and differences;

(3) the closure of a subset in $\mathcal{C}$ is in $\mathcal{C}$.

In the second step, one defines the *geometric prechains* (relative to $\mathcal{C}$). Basically, they can be represented as $\sum_\alpha m_\alpha S_\alpha$, where each S_α is a closed subset from $\mathcal{C}$ (with a fixed orientation) and m_α is an integer (the "multiplicity of S_α"). A geometric chain

is an equivalence class of geometric prechains with respect to a natural equivalence relation. (For details, see [G].)

In this paper we will use sub-analytic chains associated with our complex analytic manifolds: if M is a complex analytic manifold, and we fix a real analytic structure on M, then the class of all sub–analytic subsets of M form the good class of subsets $\mathcal{C}$.

II.1.3. **The definition of the (geometric) chains.** Now, we define our chains. In some of the definitions we follow closely the paper [GM] of Goresky and MacPherson (where the case of the P.L. geometric chains is considered). We will regard our manifold X as a pseudomanifold of dimension $2n$, where n is the complex dimension of X. We consider its natural stratification $X_{2n-2p+1}=X_{2n-2p}=Y^p$ for $p\geq 1$, hence $\Sigma=Y$. Similarly as above $\tilde{Y}^0=Y^0=X$.

By [H2], X admits a (canonical) sub–analytic triangulation, compatible with the stratification. In fact, any two sub–analytic triangulation admits a common refinement (see [H2] 2.4).

For the next definitions, we fix an integer $p\geq 0$. Then $\tilde{Y}^p$ is again a pseudomanifold. This stratification will be denoted by $\{\tilde{Y}^p_{2n-2p-2r}\}_{r\geq 0}$.

If T is a sub–analytic triangulation, let $C^T_*(\tilde{Y}^p)$ be the chain complex of simplicial chains of $\tilde{Y}^p$ with respect to T. By definition, a chain of $\tilde{Y}^p$ is an element of $C^T_*(\tilde{Y}^p)$ for some sub–analytic triangulation T, however one identifies two chains $c\in C^T_*$ and $c'\in C^{T'}_*$ if their canonical images in $C^{T''}_*$ coincide, for some common refinement T'' of T and T'. The group of all chains is denoted by $C_*(\tilde{Y}^p)$. Obviously, there is a natural boundary operator ∂ which makes $C_*(\tilde{Y}^p)$ a complex.

Notice that $C_*(\tilde{Y}^p)$ is exactly the group of sub-analytic geometric chains discussed in II.1.2. Indeed, if we fix an arbitrary closed sub-analytic subset S of $\tilde{Y}^p$, then by [H2] there is a sub-analytic triangulation T which makes S an element of $C^T_*(\tilde{Y}^p)$ (with all multiplicities one).

If $\xi\in C^T_k$, then the support $|\xi|$ of ξ is the union of the closures of those k–simplices σ for which the coefficient of ξ is non–zero. Actually, the support of ξ is independent on T, and it is a k–dimensional sub-analytic subset. We emphasize (even if this is clear in most of the cases, since X is compact), that in all our discussions, we will deal with chains with *compact* supports. (See also [B], I.) This remark is essential, when we will consider the sheaf-versions of our complexes.

The homology of the complex $C_*(\tilde{Y}^p)$ is the usual homology $H_*(\tilde{Y}^p)$ (cf. [G], page 155). Moreover, for any triangulation T, the natural morphism $C^T_*(\tilde{Y}^p)\to C_*(\tilde{Y}^p)$ is a quasi-isomorphism. Indeed, the triangulation homeomorphism $t:|K|\to\tilde{Y}^p$ from a simplicial complex $|K|$ to $\tilde{Y}^p$ identifies the simplicial homology of $|K|$ and the usual homology of $\tilde{Y}^p$.

II.2. **"Dimensionally transverse" chains.**

We introduce the following double complex of *dimensionally transverse* chains in $\tilde{Y}^p, p\geq 0$. We say that a chain $\xi\in C_k(\tilde{Y}^p)$ is *dimensionally transverse*, if

a) $\dim(|\xi|) \cap \tilde{Y}^p_{2n-2p-2r} \leq k - 2r$, for any $r > 0$; and
b) $\dim(|\partial\xi|) \cap \tilde{Y}^p_{2n-2p-2r} \leq k - 1 - 2r$, for any $r > 0$.

The subgroup of $C_*(\tilde{Y}^p)$ consisting of the *dimensionally transverse* chains is denoted by $C^{\pitchfork}_*(\tilde{Y}^p)$. The boundary operator ∂ maps dimensionally transverse chains in dimensionally transverse chains, hence defines a complex with $\partial^2 = 0$.

The reader, familiar with intersection homology, immediatelly will realize that the complex $C^{\pitchfork}_*(\tilde{Y}^p)$ is exactly the complex $IC^{\bar{0}}_*(\tilde{Y}^p)$ of intersection chains corresponding to the zero perversity (see [GM]).

II.2.1. **Lemma.** *The natural inclusion $j : (C^{\pitchfork}_*(\tilde{Y}^p), \partial) \to (C_*(\tilde{Y}^p), \partial)$ is a quasi-isomorphism. In particular, the homology of $(C^{\pitchfork}_*(\tilde{Y}^p), \partial)$ is $H_*(\tilde{Y}^p)$.*

Proof. This follows from section 4.3 of [GM] (because the Poincaré map $H^{2n-2p-k}(\tilde{Y}^p) \to H_k(\tilde{Y}^p)$ is an isomorphism, provided by the smoothness of $\tilde{Y}^p$). Actually, since $\tilde{Y}^p$ is smooth, and the intersection homology group is independent on the stratification (3.2 in [GM]), all the intersection homology groups are the same and equal to the usual homology. □

II.2.2. **The intersections with the strata.** We want to define the intersection $\xi \cap Y_\alpha \in C^{\pitchfork}_{k-2}(Y_{\alpha_1,\dots,\alpha_p} \cap Y_\alpha)$ for any chain $\xi = \xi_{\alpha_1<\dots<\alpha_p} \in C^{\pitchfork}_k(Y_{\alpha_1,\dots,\alpha_p})$, where $\alpha \notin \{\alpha_1, \dots, \alpha_p\}$. (Here, if $p = 0$ then $Y_{\alpha_1,\dots,\alpha_p}$ denotes X.) First notice the following fact (cf. [GM], page 138).

II.2.3. **Fact.** *If $C \subset \tilde{Y}^{p+1}$ is a (k–2)–dimensional sub-analytic subset of $\tilde{Y}^{p+1}$ and if $D \subset C$ is a (k–3)–dimensional sub-analytic subset, then there is a one-to-one correspondence between chains $\beta \in C_{k-2}(\tilde{Y}^{p+1})$ such that $|\beta| \subset C$, $|\partial\beta| \subset D$, and between homology classes $\tilde{\beta} \in H_{k-2}(C, D)$. Furthermore, $\partial\beta$ corresponds to the class $\partial_*(\tilde{\beta})$ in $H_{k-3}(D)$ under the connecting homomorphism $\partial_* : H_{k-2}(C, D) \to H_{k-3}(D)$.*

In the definition of the intersection $\xi \cap Y_\alpha$, we face the problem of the notion of an "intersection multiplicity". This is clarified in the work of Lefschetz, however we will work in the spirit of [GM]. It is clear that $|\xi| \cap Y_\alpha$ has dimension $\leq k - 2$, and this intersection satisfies the transversality restrictions of the new space. We have to determine the coefficients of the simplices which support the intersection.

Using the above Fact, the *intersection* $\xi \mapsto \xi \cap Y_\alpha$ is completely determined by the following composition:

$$\begin{array}{c}
H_k(|\xi|, |\partial\xi|) \\
\approx \uparrow \quad \cap[Y_{\alpha_1,\dots,\alpha_p}] \\
H^{2n-2p-k}(Y_{\alpha_1,\dots,\alpha_p} - |\partial\xi|, Y_{\alpha_1,\dots,\alpha_p} - |\xi|) \\
\downarrow \quad i^* \\
H^{2n-2p-k}(Y_{\alpha_1,\dots,\alpha_p} \cap Y_\alpha - |\partial\xi|, Y_{\alpha_1,\dots,\alpha_p} \cap Y_\alpha - |\xi|) \\
\approx \downarrow \quad \cap[Y_{\alpha_1,\dots,\alpha_p} \cap Y_\alpha]
\end{array}$$

$$H_{k-2}(|\xi|\cap Y_\alpha, |\partial\xi|\cap Y_\alpha).$$

Above, the first map is the (inverse) of the cap product with the fundamental class of $Y_{\alpha_1,\dots\alpha_p}$, as it is presented in the Appendix of [GM], page 162; the second map is the restriction i^*, where i is the natural inclusion; and finally, the third map is again the cap product with the fundamental class of $Y_{\alpha_1,\dots\alpha_p}\cap Y_\alpha$. The cap products are isomorphisms (cf. [*loc.cit.*]).

Then the *intersection* $\xi\mapsto\xi\cap Y_\alpha$ is defined as follows; ξ *determines an element in* $H_k(|\xi|,|\partial\xi|)$ *via Fact, and the image of that element by the above composition determines* $\xi\cap Y_\alpha$, again by the above Fact.

Remark. It is important to note that $|\xi|\cap Y_\alpha$ is not necessarily equal to $|\xi\cap Y_\alpha|$ since the chains are not necessarily transversal in the differential sense (cf. II.3.7).

Next, we define a (boundary) operator $\cap : C_k^{\pitchfork}(\tilde Y^p)\to C_{k-2}^{\pitchfork}(\tilde Y^{p+1})$. If $\xi\in C_k^{\pitchfork}(\tilde Y^p)$, we write $\xi^k_{\alpha_1<\cdots<\alpha_p}$ for the corresponding components of ξ corresponding to the decomposition of $\tilde Y^p$ (the index k emphasizes the dimension, and sometimes it is omitted). Then we define:

$$\cap(\xi^k_{\alpha_1<\cdots<\alpha_p}) = \sum_{\alpha_1<\cdots<\alpha_i<\alpha<\cdots<\alpha_p} (-1)^{k+i}\ \xi^k_{\alpha_1<\cdots<\alpha_p}\cap Y_\alpha.$$

II.2.4. **Lemma.** *a)* $\cap^2=0$ *and b)* $\partial\cap+\cap\,\partial=0$.

Proof. a) Let $\alpha<\beta$ and $\alpha,\beta\notin\{\alpha_1,\dots,\alpha_p\}$, where $\alpha_1<\cdots<\alpha_i<\alpha<\alpha_{i+1}<\cdots<\alpha_{i+j}<\beta<\cdots<\alpha_p$. ¿From the properties of the intersection (see, e.g. [GM], page 144), one obtains that $(\xi_{\alpha_1<\cdots<\alpha_p}\cap Y_\alpha)\cap Y_\beta=(\xi_{\alpha_1<\cdots<\alpha_p}\cap Y_\beta)\cap Y_\alpha$. Then the sum of the following two contributions in $\cap^2$ is zero:

$$\cap_{Y_\beta}(\cap_{Y_\alpha}\xi^k_{\alpha_1<\ldots<\alpha_p}) = (-1)^{k+i}\cap_{Y_\beta}(\xi^k_{\alpha_1<\cdots<\alpha_p}\cap Y_\alpha) = (-1)^{j+1}(\xi^k_{\alpha_1<\cdots<\alpha_p}\cap Y_\alpha)\cap Y_\beta$$

$$\cap_{Y_\alpha}(\cap_{Y_\beta}\xi^k_{\alpha_1<\ldots<\alpha_p}) = (-1)^{k+i+j}\cap_{Y_\alpha}(\xi^k_{\alpha_1<\cdots<\alpha_{i+j}<\cdots<\alpha_p}\cap Y_\beta) = (-1)^{j}(\xi^k_{\alpha_1<\cdots<\alpha_p}\cap Y_\beta)\cap Y_\alpha.$$

b) follows from the definition of $\cap$ and ∂, including the sign $(-1)^k$ when $\dim|\xi|=k$. □

The fact that the homology of $(C_*^{\pitchfork}(\tilde Y^p),\partial)$ is exactly $H_*(\tilde Y^p)$ (cf. Lemma II.2.1) and part (b) of II.2.4 show that $\cap$ induces an operator $H_*(\tilde Y^p)\to H_{*-2}(\tilde Y^{p+1})$.

II.2.5. **Lemma.** *For any* $p\geq 0$, *the operator* $H_*(\tilde Y^p)\to H_{*-2}(\tilde Y^{p+1})$ *induced by* $\cap$ *is*

$$\cap[\xi^k_{\alpha_1<\cdots<\alpha_p}] = \sum_{\alpha_1<\cdots<\alpha_i<\alpha<\cdots<\alpha_p} (-1)^{k+i}\ [\xi^k_{\alpha_1<\cdots<\alpha_p}]\cap[Y_\alpha]$$

where $\cap[Y_\alpha]$ *denotes the homological Gysin map, or transfer map* $i_!$, *where* $i: Y_{\alpha_1,\dots,\alpha_p}\cap Y_\alpha\hookrightarrow Y_{\alpha_1,\dots,\alpha_p}$ *is the natural inclusion. More precisely,* $i_!=PD\circ i^*\circ PD$, *where* PD *denotes the Poincaré Dualities in the corresponding spaces (cf. e.g. [Br], page 368).*

Proof. The result follows from the definition of the intersection $\xi\cap Y_\alpha$. □

II.2.6. **The Poincaré duality map.** We will need later the Poincaré isomorphism between homology of Y and cohomology of X with support in Y, so we adopt here to our situation the construction of [GM], pages 139-140, inspired by the classical construction which provides the Poincaré Duality for manifolds (see e.g. [Br], page 338).

Similarly as above, for any triangulation T, compatible with the stratification, one can consider the chain complex of simplicial cochains $(C_T^*(\tilde{Y}^p), \delta)$ of $\tilde{Y}^p$. Here $C_T^i(\tilde{Y}^p) = Hom(C_i^T(\tilde{Y}^p), \mathbb{Z})$. Let T' be the first barycentric subdivision of T, and let $\hat{\sigma}$ denote the barycentre of the simplex $\sigma \in T$. Let T_i be the i-skeleton of T, thought of as a subcomplex of T'. It is spanned by all vertices of $\hat{\sigma}$ such that $\dim \sigma \leq i$. Let D_i be the i-coskeleton spanned, as a subcomplex of T', by all the vertices $\hat{\sigma}$ such that $\dim \sigma \geq i$. There are canonical simplex preserving deformation retracts:

II.2.7. $$X - |T_i| \to |D_{i+1}| \quad \text{and} \quad X - |D_{i+1}| \to |T_i|.$$

Now, identify $C_T^i(X)$ with $\oplus_{\dim \sigma = i} H^i(\sigma, \partial\sigma) = H^i(|T_i|, |T_{i-1}|)$, and define $pd : C_T^i(X) \to C_{2n-i}^{T'}(X)$ by the following composition:

$$\begin{array}{c}
H^i(|T_i|, |T_{i-1}|) \\
\downarrow \ \cap[X] \\
H_{2n-i}(X - |T_{i-1}|, X - |T_i|) \\
\approx \downarrow \ \text{(deformation retract)} \\
H_{2n-i}(|D_i|, |D_{i+1}|) \\
\downarrow \\
H_{2n-i}(|T'_{2n-i}|, |T'_{2n-i-1}|)
\end{array}$$

Actually, $H_{2n-i}(|T'_{2n-i}|, |T'_{2n-i-1}|) = C_{2n-i}^{T'}(X)$, but we have even something more. Since the image of any chain by this composition is supported by union of $|D_i|$'s, and any $|D_i|$ is dimensionally transverse, one obtains a homomorphism $pd : C_T^i(X) \to C_{2n-i}^{T',\pitchfork}(X)$. Similarly, for any $Y_{\alpha_1,\dots,\alpha_p}$, one can define a homomorphism:

$$pd : C_T^i(Y_{\alpha_1,\dots,\alpha_p}) \to C_{2n-2p-i}^{T',\pitchfork}(Y_{\alpha_1,\dots,\alpha_p}).$$

By [GM] (7.2), this is a chain map:

II.2.8. $$\partial \circ pd = pd \circ \delta.$$

II.2.9. **Lemma.** *a) Fix $\alpha \notin \{\alpha_1, \dots, \alpha_p\}$. Then the following diagram is commutative:*

$$\begin{array}{ccc}
C_T^i(Y_{\alpha_1,\dots,\alpha_p}) & \xrightarrow{pd} & C_{2n-2p-i}^{T',\pitchfork}(Y_{\alpha_1,\dots,\alpha_p}) \\
\Big\downarrow i^* & & \Big\downarrow \cap \\
C_T^i(Y_{\alpha_1,\dots,\alpha_p} \cap Y_\alpha) & \xrightarrow{pd} & C_{2n-2p-i-2}^{T',\pitchfork}(Y_{\alpha_1,\dots,\alpha_p} \cap Y_\alpha)
\end{array}$$

b) *The above diagram (and II.2.8) provides at homology level a commutative diagram:*

$$\begin{array}{ccc} H^i(Y_{\alpha_1,\dots,\alpha_p}) & \xrightarrow{PD} & H_{2n-2p-i}(Y_{\alpha_1,\dots,\alpha_p}) \\ \big\downarrow i^* & & \big\downarrow \cap \\ H^i(Y_{\alpha_1,\dots,\alpha_p}\cap Y_\alpha) & \xrightarrow{PD} & H_{2n-2p-i-2}(Y_{\alpha_1,\dots,\alpha_p}\cap Y_\alpha) \end{array}$$

where the horizontal maps are the Poincaré Duality isomorphisms (cf. also with II.2.5).

Proof. Use the definition of $\cap$ and *pd* and II.2.7. □

II.3. **The chain correspondence $\Pi_\star^{-1}$.**

II.3.1. **Triangulation of** Z . Let us indicate quickly that the real blow up $\Pi : Z \to X$ of Y in X can be realized as a semi-analytic subset of a real vector space. Since X is compact, there is no difference between bounded semi-analytic and subanalytic simplices [H2, Def. 3.3, page 179]. The variety X is locally, analytically, isomorphic to a ball in $\mathbb{C}^n$, such that the variety Y_J is defined by $z_1 \cdot \ldots \cdot z_j = 0$ where $j = |J|$, then the variety Z is isomorphic to the semi-analytic subset of a real vector space: $T^n \times (\mathbb{R}^+)^n$ with the the product structure of $(\mathbb{R}^+)^n$ and the n-dimensional torus. The projection Π is written, in the coordinates θ_k on the torus, ρ_k on the reals and z_k on $\mathbb{C}^n$ for $k = 1, ..., n$, as $z_k \circ \Pi = \rho_k\ e^{i\theta_k}$. We have

$$\Pi^{-1}(Y_J - Y^{j+1}) \simeq T^j \times T^{n-j} \times (\mathbb{R}^*)^{n-j} \simeq T^j \times (\mathbb{C}^*)^{n-j}$$

where we use in the last isomorphism the correspondence defined by $z_k = \rho_k e^{i\theta_k}, k = j+1, ..., n$ to describe $Y_J - Y^{j+1}$.

Define the strict transform of a simplex Δ on $\tilde{Y}^r$ as the closure of the inverse image in Z of the complement of Y^{r+1} in Δ.

Let Δ_α be a semi-analytic (resp. sub-analytic) simplex contained in the domain of a chart on X and defined by a set of equalities and inequalities of analytic functions $f_{\alpha,\lambda}(x., y.)_{\lambda\in\Lambda}$ in the real coordinates $(x., y.)$ of $\mathbb{C}^n$; its inverse image is obtained by replacing $(x., y.)$ with $\rho_.(cos(\theta.), sin(\theta.))$ in the defining equations. The strict transform of the simplices Δ_α in X are semi-analytic subsets of Z (resp. sub-analytic).

They define in Z a decomposition in a finite system of sub-analytic sets, so that we can use Hironaka's existence theorem [H2, page 180] to define a triangulation by sub-analytic simplices, subordinated to this decomposition.

II.3.2. In this subsection we present our *main topological tool*: a correspondence denoted $\Pi_\star^{-1}$ which gives closed chains in Z, respectively in its boundary ∂Z, as "strict inverse" of chains on X. But first we have to fix some orientation conventions regarding the fibers $\Pi^{-1}(y^o)$ for different points $y^o \in Y$.

We denote the oriented boundary of a disc in the complex plane by $S^1 = \partial D$ (where we consider the natural orientation as a boundary). For any point $y^o \in Y^1 - Y^2$, the circle $\Pi^{-1}(y^o)$ appears as the boundary of the complement of a disc in $\mathbb{C}$, hence it is $-S^1$. If $y^o \in Y^p - Y^{p+1}$, the situation is similar. In a local model, when Y is defined by $\{y_1 \dots y_p = 0\} \subset \mathbb{C}^p$ where y^o stays for the origin. Let U be defined by $\{y \mid \min |y_i| \leq 1\} \subset \mathbb{C}^p$ and the component Y_{α_i} by $y_i = 0$. Then $\Pi^{-1}(y^o)$, set-theoretically, is the tori $S^1_1 \times \ldots \times S^1_p$, where S^1_i is defined by $\{y_i : |y_i| = 1\}$. Then, we fix the orientation

of $\Pi^{-1}(y^o)$ as given by the product orientation $(-S^1_1) \times \cdots \times (-S^1_p)$. Moreover, if σ is a contractible C^∞-submanifold in $Y_{\alpha_1} \cap \cdots \cap Y_{\alpha_p} - Y^{p+1}$ $(\alpha_1 < \cdots < \alpha_p)$, then on the set-theoretical inverse image $\Pi^{-1}(\sigma)$ we define the product orientation $\sigma \times (-S^1_{\alpha_1}) \times \cdots \times (-S^1_{\alpha_p})$.

With this notations, the following holds.

II.3.3. **Lemma.** *Fix indices $\alpha_1 < \cdots < \alpha_i < \alpha < \alpha_{i+1} < \cdots < \alpha_p$ and let $\sigma \subset Y_{\alpha_1} \cap \cdots \cap Y_{\alpha_p}$ be an oriented $\mathbb{C}^\infty$ sub-manifold with boundaries such that σ and $\partial\sigma$ intersect Y_α transversally, but $\sigma \cap Y_\beta = \varnothing$ for any $\beta \notin \{\alpha, \alpha_1, \ldots, \alpha_p\}$. (The C^∞ transversal intersection is considered in $Y_{\alpha_1} \cap \cdots \cap Y_{\alpha_p}$.) Then*

$$\partial(\Pi^{-1}\sigma) = \Pi^{-1}(\partial\sigma) + (-1)^{\dim\sigma + i}\Pi^{-1}(\sigma \cap Y_\alpha).$$

Proof. In a neighbourhood of an intersection point $p \in (\sigma \cap Y_\alpha)$ (respectively$(\partial\sigma) \cap Y_\alpha$), σ has a product structure $D \times T$ where T is a ball (respectively a half ball) in $Y_\alpha \cap Y_{\alpha_1} \cap \cdots \cap Y_{\alpha_p}$, and D is a real 2-disc transversal to Y_α. We denote the boundary of D by ∂D. Let $D_\eta = D - \{$small open disc of radius η and origin $0\}$; i.e. $\partial D_\eta = \partial D - S^1_\eta$. Then we have the diffeomorphism $\Pi^{-1}(D \times T) \approx \Pi^{-1}(D_\eta \times T)$. Since $\Pi^{-1}(D_\eta \times T) = D_\eta \times T \times \Omega$, where $\Omega = (-1)^p S^1_{\alpha_1} \times \cdots \times S^1_{\alpha_p}$, we get: $\partial\Pi^{-1}(D \times T) = \partial\Pi^{-1}(D_\eta \times T) = \partial D \times T \times \Omega - S^1_\eta \times T \times \Omega + D_\eta \times \partial T \times \Omega =$
$= (-1)^{\dim T + i} T \times (-1)^{p+1} S^1_{\alpha_1} \times \cdots \times S^1_\alpha \times \cdots \times S^1_{\alpha_p} + \partial(D \times T) \times \Omega$
$= (-1)^{i + \dim\sigma}\Pi^{-1}(\sigma \cap Y_\alpha) + \Pi^{-1}\partial\sigma.$ □

II.3.4. Next, we want to lift an arbitrary sub-analytic chain. First notice that for each sub-analytic subset $S \subset \tilde{Y}^r$, the strict transform $\tilde{S} := cl(\Pi^{-1}(S - Y^{r+1}))$ (i.e. the closure of the inverse image of the complement of Y^{r+1} in S) is a sub-analytic subset of Z. Indeed, the strict transform can be constructed using a real analytic isomorphism and the permitted operations listed in II.1.2 (2-3) (cf. also with II.3.1).

Fix again the integer $p \geq 0$, and for any chain $\xi \in C^\pitchfork_k(\tilde{Y}^p)$ we want to construct a chain $\Pi^{-1}_p(\xi)$ in $C_{k+p}(\partial Z)$ if $p \geq 1$, respectively in $C_k(Z)$ if $p = 0$. By the above remark, the inverse image $\Pi^{-1}(|\xi|)$ of the support of ξ is a $(k+p)$–dimensional sub–analytic subset of Z. Similarly as above, in the construction we will use Fact II.2.3.

Like in the case of Leray's cohomological residue, we have the following morphism of relative homology:

$$\Pi^{-1}_{p,rel} : H_k(|\xi|, |\partial\xi| \cup (|\xi| \cap Y^{p+1})) \longrightarrow H_{k+p}(\Pi^{-1}(|\xi|), \Pi^{-1}(|\partial\xi|) \cup \Pi^{-1}(|\xi| \cap Y^{p+1})).$$

Since $\Pi^{-1}(|\xi|) \to |\xi|$ is an oriented fiber bundle over $|\xi| - Y^{p+1}$, with fibers $(S^1)^p$, the classical spectral sequence argument leads to an isomorphism. Indeed, we use deformation retract tubular neighbourhoods to thicken $|\xi| \cap Y^{p+1}$ and its inverse image $\Pi^{-1}(|\xi| \cap Y^{p+1})$, then the excision theorem to reduce to the fiber bundle case and show that the relative morphism above is an isomorphism.

Next, we use the morphism

$$i : H_k(|\xi|, |\partial\xi|) \longrightarrow H_k(|\xi|, |\partial\xi| \cup (|\xi| \cap Y^{p+1}))$$

induced by the inclusion $(|\xi|, |\partial\xi|) \longrightarrow (|\xi|, |\partial\xi| \cup (|\xi| \cap Y^{p+1}))$. Notice that i is an isomorphism since $\mathrm{codim}|\xi| \cap Y^{p+1}$ is 2 in $|\xi|$ and in the long homology exact sequence

of the pair $H_i(|\partial\xi| \cup (|\xi| \cap Y^{p+1}), |\partial\xi|) = H_i(|\xi| \cap Y^{p+1}, |\partial\xi| \cap Y^{p+1}) = 0$ if $i = k$ or $k-1$.

Now, the image of ξ, via the composed map $\Pi^{-1}_{p,rel} \circ i$, determines completely $\Pi^{-1}_p(\xi)$ (via Fact). By definition, this is the application $\xi \mapsto \Pi^{-1}_p(\xi)$. Sometimes the index p will be replaced by $\star$.

In a simple way, this chain $\Pi^{-1}_p(\xi)$ can be defined as follows. Write ξ as a finite sum $\sum a_\sigma \sigma$, where σ are the simplices in the support of ξ with non-zero coefficients. Let σ^0 be $\sigma - Y^{p+1}$. Then the closure $cl(\Pi^{-1}(\sigma^0))$ of $\Pi^{-1}(\sigma^0)$ is a $(k+p)$–dimensional sub–analytic set. Then define $\Pi^{-1}_p(\xi)$ by $\sum\ a_\sigma\, cl(\Pi^{-1}(\sigma^0))$.

The next result generalizes the above lemma.

II.3.5. **Proposition.** *Fix the integer $p \geq 0$ and indices $\alpha_1 < \cdots < \alpha_p$. For any $\alpha \notin \{\alpha_1, \ldots, \alpha_p\}$, set $i(\alpha) = i$ if $\alpha_1 < \cdots < \alpha_i < \alpha < \alpha_{i+1} < \cdots < \alpha_p$. Then for any $\xi \in C^{\pitchfork}_k(Y_{\alpha_1,\ldots,\alpha_p})$:*

$$\partial(\Pi^{-1}_p \xi) = \Pi^{-1}_p(\partial\xi) + \sum_\alpha\ (-1)^{k+i(\alpha)}\ \Pi^{-1}_{p+1}(\xi \cap Y_\alpha).$$

Above, the sum is over all α with $\alpha \notin \{\alpha_1, \ldots, \alpha_p\}$. The equality is considered in $C_*(\partial Z)$ if $p \geq 1$, and in $C_*(Z)$ if $p = 0$.

Proof. The contribution $\Pi^{-1}_p(\partial\xi)$ is clear. Next we want to determine the coefficients of the simplices which lie in $\Pi^{-1}(|\xi| \cap Y_\alpha)$. It is enough to work modulo $Y_\alpha \cap (\cup_\beta Y_\beta)$, where the union is over all $\beta \notin \{\alpha, \alpha_1, \ldots, \alpha_p\}$, since it intersects $|\xi|$ in codimension 4 and its inverse image intersects $\Pi^{-1}(|\xi|)$ in codimension 2.

Consider the composition:

$$\begin{array}{c} H_{k+p}(\Pi^{-1}(|\xi|), \Pi^{-1}(|\partial\xi|) \cup \Pi^{-1}(|\xi| \cap Y^{p+1})) \\ \downarrow\ \partial \\ H_{k+p-1}(\Pi^{-1}(|\partial\xi| \cup (|\xi| \cap Y^{p+1})), \Pi^{-1}(|\partial\xi| \cup (|\xi| \cap \cup_\beta Y_\beta))) \\ \approx \downarrow\ e \\ H_{k+p-1}(\Pi^{-1}(|\xi| \cap Y_\alpha), \Pi^{-1}(|\partial\xi| \cap Y_\alpha) \cup \Pi^{-1}(|\xi| \cap Y_\alpha \cap \cup_\beta Y_\beta))). \end{array}$$

Again, the union $\cup_\beta$ is over all $\beta \notin \{\alpha, \alpha_1, \ldots, \alpha_p\}$. If $A = \Pi^{-1}(|\xi|)$, $B = \Pi^{-1}(|\partial\xi| \cup (|\xi| \cap \cup_\beta Y_\beta)))$, and $C = \Pi^{-1}(|\xi| \cap Y_\alpha)$, then the first map is the boundary operator $H_{k+p}(A, B \cup C) \to H_{k+p-1}(B \cup C, B)$, and the second is the excision isomorphism $H_{k+p-1}(B \cup C, B) \to H_{k+p-1}(C, B \cap C)$.

The composed map $e \circ \partial$ and the map $\xi \mapsto \xi \cap Y_\alpha$ are connected by the following diagram, which commutes up to sign:

$$\begin{array}{ccc} H_k(|\xi|, |\partial\xi|) & \xrightarrow{\Pi^{-1}_p} & H_{k+p}(\Pi^{-1}(|\xi|), \Pi^{-1}((|\partial\xi|) \cup (|\xi| \cap Y^{p+1}))) \\ \downarrow\ \cap Y_\alpha & & \downarrow\ e \circ \partial \\ H_{k-2}(|\xi| \cap Y_\alpha, |\partial\xi| \cap Y_\alpha) & \xrightarrow{\Pi^{-1}_{p+1}} & H_{k+p-1}(\Pi^{-1}(|\xi| \cap Y_\alpha), \Pi^{-1}((|\partial\xi| \cap Y_\alpha) \cup (|\xi| \cap Y_\alpha \cap \cup_\beta Y_\beta))) \end{array}$$

The commutativity of the diagram (up to a sign), basically comes from the fact that for a manifold with boundary, the Lefschetz duality identifies the boundary operator in homology with the restriction map (to the boundary) in cohomology (see e.g. [Br], page 357). The above sign is universal, depends only on the orientation conventions. Therefore, it can be determined using C^∞-transversal chains, as in lemma II.3.3. $\square$

The above proposition together with the definition of the operator $\cap : C_k^{\pitchfork}(\tilde{Y}^p) \to C_{k-2}^{\pitchfork}(\tilde{Y}^{p+1})$, have the following corollary.

II.3.6. **Corollary.** $\partial\Pi_\star^{-1} = \Pi_\star^{-1}(\partial + \cap)$.

II.3.7. **Example.** Let $\xi \in C_*^{\pitchfork}(X)$. In general, the three sets $\Pi^{-1}(|\xi| \cap Y)$ (here Π^{-1} denotes the set-theoretical inverse image), $|\Pi_\star^{-1}(\xi)| \cap \partial Z$ and $|\partial\Pi_\star^{-1}(\xi)|$ are all distinct. To see this, set $X = \mathbb{C}^2$, $Y = \{0\}\times\mathbb{C}$ ($Y^2 = \varnothing$). Fix coordinates $z_j = x_j + iy_j$ ($j = 1, 2$) in $\mathbb{C}^2$. Let ξ be a chain with support $(x_1 - 1)^2 + y_1^2 + x_2^2 - 1 = y_2 = 0$, and coefficient one.

Then $|\xi|$ is a 2-dimensional real sphere with $|\xi| \cap Y = (0, 0)$, hence $\Pi^{-1}(|\xi| \cap Y) = S^1$. On the other hand, the intersection chain $\xi \cap Y = 0$ (even if $|\xi| \cap Y \neq \varnothing$). Actually, $\xi \cap Y$ is the chain supported by $|\xi| \cap Y$ with coefficient the intersection multiplicity of $|\xi|$ and Y, which is zero. Therefore, $\partial\Pi_\star^{-1}(\xi) = \Pi_\star^{-1}\partial\xi = 0$, hence $|\partial\Pi_\star^{-1}(\xi)| = \varnothing$. Finally, the reader is invited to verify that $|\Pi_\star^{-1}(\xi)| \cap \partial Z$ is a *half circle* in $S^1 = \Pi^{-1}(0, 0)$.

II.3.8. **Remark.** In the above discussion, corresponding to the integer $p = 0$, one can replace the space X by the (compact) tubular neighbourhood $U = U_\alpha = \Pi(Z_\alpha)$ (cf. I.1.2). This means that the complex $C_*(X)$ is replaced by $C_*(U)$, and $C_*^{\pitchfork}(X)$ by $C_*^{\pitchfork}(U)$. Notice that $\cap : C_*^{\pitchfork}(U) \to C_{*-2}^{\pitchfork}(\tilde{Y}^1)$ is well-defined and still satisfies $\cap\partial + \partial\cap = 0$. Moreover, one has a map $\Pi_\star^{-1} : C_*^{\pitchfork}(U) \to C_*(Z_\alpha)$, defined similarly as $\Pi_\star^{-1} : C_*^{\pitchfork}(X) \to C_*(Z)$, which satisfies $\partial\Pi_\star^{-1} = \Pi_\star^{-1}(\partial + \cap)$.

II.4. **The complex of sheaves.** $C_*(\tilde{Y}^p)$ has a natural sheafification $\mathcal{C}^*_{\tilde{Y}^p}$ which is fine ([GM2] page 97, or [B] page 33). Actually, the complex $\mathcal{C}^*_{\tilde{Y}^p}$ is quasi-isomorphic to the dualizing complex $\mathcal{D}^*_{\tilde{Y}^p}$ (cf. [GM2] page 97, or [B] page 33).

What we will need in this paper, is a different construction, we will sfeafify the dual complex, and we will obtain a quasi-isomorphism with $\mathbb{Z}_{\tilde{Y}^p}$.

For any open set $V \subset \tilde{Y}^p$, let $C_k(V, \tilde{Y}^p)$ be the subgroup of chains $\xi \in C_k(\tilde{Y}^p)$ with compact support $|\xi|$ in V. Obviously, for any open pair $V \subset W \subset \tilde{Y}^p$, there is a natural inclusion $C_k(V, \tilde{Y}^p) \to C_k(W, \tilde{Y}^p)$. Now, define the dual $C^k(V, \tilde{Y}^p)$ by $Hom_{\mathbb{Z}}(C_k(V, \tilde{Y}^p), \mathbb{Z})$. Then for any $V \subset W$ as above, the "restriction" $C^k(W, \tilde{Y}^p) \to C^k(V, \tilde{Y}^p)$ defines a presheaf $\mathcal{C}^k(\tilde{Y}^p)$ on $\tilde{Y}^p$, satisfying the condition (S2) (i.e. it is "conjuctive" in the terminology of [Br]). Let $\bar{\mathcal{C}}^k(\tilde{Y}^p)$ be the associated sheaf with global sections $\bar{C}^k(\tilde{Y}^p)$; and let $C_0^k(\tilde{Y}^p)$ be the subgroups of elements of $C^k(\tilde{Y}^p)$ with empty support. Then

$$0 \to C_0^k(\tilde{Y}^p) \to C^k(\tilde{Y}^p) \to \bar{C}^k(\tilde{Y}^p) \to 0$$

is exact (cf. [Br] page 22). Moreover, $C^*(\tilde{Y}^p)$ and $C_0^*(\tilde{Y}^p)$ form complexes, and $H^*(C_0^*(\tilde{Y}^p))=0$ (by subdivision argument, see [Br2], page 26 in the case of singular chains).

Therefore, the complexes $C^*(\tilde{Y}^p)$ and $\bar{C}^*(\tilde{Y}^p)$ are quasi-isomorphic. On the other hand, for any open V, there is an augmentation map $C_0(V,\tilde{Y}^p)\to\mathbb{Z}$ which give rise to a resolution $0\to\mathbb{Z}_{\tilde{Y}^p}\to\bar{\mathcal{C}}^*(\tilde{Y}^p)$. The sheaf $\bar{\mathcal{C}}^k(\tilde{Y}^p)$ is a module over the ring of $\mathbb{Z}$-constructible functions on $\tilde{Y}^p$. Indeed, for any constructible function f and $\varphi\in C^k(\tilde{Y}^p)$ one can define $f\cdot\varphi\in C^k(\tilde{Y}^p)$ by $(f\cdot\varphi)(\sigma)=f(\hat{\sigma})\varphi(\sigma)$, where $\hat{\sigma}$ is the barycenter of σ. This shows that the above resolution is a resolution of fine sheaves.

III. Double complexes and their spectral sequences

We construct in this section various homological complexes giving rise to various homology groups with their weight filtration. The cases $(Y,\varnothing)$ and (X,Y) are well-known, but we will need them in the construction of the double complex of ∂U, so we include them in our presentation as well. The new result is the construction of the complexes of the pairs $(X-Y,\varnothing)$, $(X,X-Y)$ and $(\partial U,\varnothing)$, in which cases we will use the dimensionally transverse cycles. The most involved case is ∂U, which is separated in section VI. In the next paragraphs, as an introductory guide, we will stress the case $X-Y$.

The *main new object is homological double complex* of $X-Y$:

$$A_{s,t}(X-Y):=C^{\pitchfork}_{t+2s}(\tilde{Y}^{-s}),\, s\leq 0, t+2s\geq 0$$

with $D=\partial+\cap$ as the differential of the total complex $\mathrm{Tot}_*(A_{**}(X-Y))$.

Our aim, reformulated for the case of $X-Y$, is to prove that the homology of the above total complex is the homology of $X-Y$, and its weight filtration provides Deligne's weight filtration on $H_*(X-Y,\mathbb{Q})$. More precisely:

III.0. **Theorem.** *Let $Y=\bigcup_{i\in I}Y_i$ be a NCD in a smooth proper algebraic variety X over $\mathbb{C}$ and $\Pi:Z\to X$ the projection as above. The generalized Leray inverse image defines a quasi-isomorphism*

$$\Pi_\star^{-1}:\mathrm{Tot}_*(A_{**}(X-Y))\longrightarrow C_*(Z).$$

In other words, a collection of dimensionally transversal geometric chains $\{c_r\}_{r\geq p}$, where c_r is a chain on $\tilde{Y}^r$ satisfying $\partial c_p=0$ and $\partial c_{r+1}=c_r\cap\tilde{Y}^{r+1}$ for any $r\geq p$ ($p\geq 0$), defines a cycle $c=\sum_{r\geq p}\Pi_r^{-1}c_r$ in Z. Moreover, their homology classes generate $H_*(Z,\mathbb{Q})$.

Geometrically this means the following: the highest nontrivial weight in $H_k(X-Y,\mathbb{Q})$ is $-k$, and for any $p\geq 0$, $W_{-k-p}H_k(X-Y,\mathbb{Q})$ is generated by classes of cycles e, with a chain decomposition $e=\sum_{r\geq p}e_r$ such that each e_r is the closure of a "fiber bundle" over a c_r-Y^{r+1} with fiber $(S^1)^r$, where c_r are chains in $\tilde{Y}^r$ and c_r-Y^{r+1} is considered in Y^r. This is realized geometrically in a small tubular neighbourhood of Y^r. The fact that the collection $\{c_r\}_r$ satisfies the above condition is equivalent with the fact that $\{e_r\}_r$ fit together without boundary, forming a closed cycle.

Above we identified $H_*(C_*(Z))$ and $H_*(X-Y)$. For more details, see III.4 and V.2.

This result can be obtained by duality with Deligne's logarithmic differential complex. Here we follow a purely topological path:

1) We start with the basic Mayer–Vietoris resolution for the NC-divisor Y.

2) We introduce the relative homology of $X\,mod\,Y$ using the classical cone construction, called here *mixed cone*, in order to carry the construction with weight filtrations (a diagonalisation process for the weights is needed in order to stay in the category of mixed Hodge complexes [D] so to get mixed Hodge structures on their (co)homology by the basic result of Deligne).

3) We give a topological construction of Poincaré duality between the cohomology of Y and the homology of the pair $(X, X-Y)$.

4) The above duality extends to a duality between the cohomology of the pair (X, Y) and the homology of $X-Y$. This provides the wanted result.

Moreover we obtain the degeneration of the weight spectral sequence on the total complex of dimensionally transversal chains.

Notations and preliminary remarks. For any double complex $(A_{**}, \partial, \delta)$, we denote its total complex by $(Tot_*(A_{**}), D)$, where $Tot_k(A_{**}) = \oplus_{s+t=k} A_{st}$ and $D = \partial + \delta$. (Here the degree of ∂ is $(0,-1)$, of δ is $(-1,0)$.) The weight filtration of A_{**} is defined by $W(A_{**})_s := \oplus_{p\leq s} A_{pq}$. The homological spectral sequence associated with the weight filtration W is denoted by E^r_{**}. Recall that $E^1_{st} = H_t(A_{s*}, \partial)$ and d^1 is induced by δ. However, we will violate the weight notation on the ∞-term, and we will use Deligne's convention: on E^∞_{st} the weight is $-t$ (instead of s); i.e. image$\{H_i(Tot_*(W_s) \to H_i(Tot_*(A))\}$ is $W_{s-i} H_i(Tot_*(A))$.

The *dual double complex* of A_{**} is $B_{st} = Hom_{\mathbb{Z}}(A_{st}, \mathbb{Z})$ with the corresponding dual maps. Its weight filtration is $W(B)_s = \{\varphi \in A^* : \varphi(W(A)_{-s-1}) = 0\}$, or equivalently, $W(B)_{-s} = \oplus_{p\geq s} B_{pq}$.

For the definition and properties of "cohomological mixed Hodge complexes" see [D.III]. In the next proofs the notion of "mixed cone" will be important, this corresponds to the mapping cone in the category of mixed Hodge complexes. For details, see [D. III], page 21, or [E], page 49.

III.1. The homological double complex of Y (the homology of the NCD).

Consider the double complex $A_{s,t}(Y) := C_t(\tilde{Y}^{s+1})$ (with $s \geq 0$ and $t \geq 0$) together with the natural operators:

$$\partial : C_k(\tilde{Y}^p) \to C_{k-1}(\tilde{Y}^p) \text{ and } \; i : C_k(\tilde{Y}^p) \to C_k(\tilde{Y}^{p-1}).$$

Here i is defined as follows. If $\oplus_{\alpha_1<\cdots<\alpha_p} c_{\alpha_1,\dots,\alpha_p} \in C_k(\tilde{Y}^p)$, then $i(\oplus_\alpha c_\alpha) = \oplus_\beta d_\beta$ if:

$$d_{\alpha_1,\dots,\alpha_{p-1}} = \sum_{\alpha_1<\cdots<\alpha_i<\alpha<\cdots<\alpha_{p-1}} (-1)^{k+i}\; i_{\alpha_1,\dots,\alpha_{p-1};\alpha}(c_{\alpha_1,\dots,\alpha_i,\alpha,\dots,\alpha_{p-1}}),$$

where $i_{\alpha_1,\dots,\alpha_{p-1};\alpha}$ is the natural inclusion $Y_{\alpha_1,\dots,\alpha_i,\alpha,\dots,\alpha_{p-1}} \hookrightarrow Y_{\alpha_1,\dots,\alpha_{p-1}}$.

III.1.1. **Lemma.** *a) $i^2 = 0$, and b) $i\partial + \partial i = 0$.*

In particular, $D := i + \partial$ is a differential of the total complex $Tot_*(A_{**}(Y))$. The weight filtration $\{W(A)_s\}_s$ provides a spectral sequence over $\mathbb{Z}$. Here are some of its properties.

III.1.2. **Proposition.**
a) $E^1_{st} = H_t(\tilde{Y}^{s+1})$ and $d^1 : E^1_{st} = H_t(\tilde{Y}^{s+1}) \to H_t(\tilde{Y}^s) = E^1_{s-1,t}$ is i_* induced by i.
b) $E^r_{st} \Longrightarrow H_{s+t}(Y, \mathbb{Z})$, inducing a weight filtration on $H_*(Y, \mathbb{Z})$. Actually, $E^\infty_{st} \otimes \mathbb{Q} = Gr^W_{-t} H_{s+t}(Y, \mathbb{Q})$ (in Deligne's weight notation).
c) $E^*_{**} \otimes \mathbb{Q}$ degenerates at level two, i.e. $d^r \otimes 1_\mathbb{Q} = 0$ for $r \geq 2$.

Proof. Let $n : \tilde{Y}^p \to Y$ be the natural map. Recall that the $K^\cdot_\mathbb{Q}$ term in Deligne's cohomological mixed Hodge complex associated with the space Y is the "Mayer-Vietoris resolution" $n_*\mathbb{Q}_{\tilde{Y}^\cdot}$:

$$0 \to n_*\mathbb{Q}_{\tilde{Y}^1} \to n_*\mathbb{Q}_{\tilde{Y}^2} \to \cdots .$$

This is considered with its "bête" filtration $W_{-s}(n_*\mathbb{Q}_{\tilde{Y}^\cdot}) = \sigma_{\geq s}(n_*\mathbb{Q}_{\tilde{Y}^\cdot})$.

Consider now the dual double complex B_{**} of A_{**}. Then $(n_*\mathbb{Q}_{\tilde{Y}^\cdot}, W)$ and $(B_{**}, W) \otimes 1_\mathbb{Q}$ are quasi-isomorphic. This follows from the discussion II.4. Therefore, their spectral sequence (for $r > 0$) are isomorphic. This gives a) and b). Finally notice that by a result of Deligne, the weight spectral sequence (over $\mathbb{Q}$) of a mixed Hodge complex degenerates at rank two, which provides c). $\square$

For a different approach and proof, see [McC], where it is proved that the filtered dualizing complex $(i_*\mathcal{D}^*_Y, \tau_\leq)$ and $(i_*n_*\mathbb{Q}_{\tilde{Y}}, \sigma_\geq)$ are Verdier dual (here $i : Y \to X$ stays for the inclusion).

III.2. The homological double complex of (X, Y).

Starting with $\tilde{Y}^0 = X$, we consider now the double complex $A_{s,t}(X, Y) := C_t(\tilde{Y}^s)$ (with $s \geq 0$ and $t \geq 0$) together with the natural operators: $\partial : C_k(\tilde{Y}^p) \to C_{k-1}(\tilde{Y}^p)$ and $i : C_k(\tilde{Y}^p) \to C_k(\tilde{Y}^{p-1})$ as above. Again, $D := i + \partial$ is a differential of the total complex $Tot_*(A_{**}(X, Y))$.

If we introduce the double complex $A_{**}(X)$ of X defined by $A_{s*}(X) = C_*(X)$ if $s = 0$ and $= 0$ otherwise, then $A_{s*}(X, Y) = A_{s*}(X) \oplus A_{s-1,*}(Y)$, and $A_{**}(X, Y)$ can be interpreted as the $Cone(\tilde{i})$ of $\tilde{i} : A_{**}(Y) \to A_{**}(X)$, where $\tilde{i}|A_{s*}(Y) = 0$ if $s \neq 0$, and $\tilde{i}|A_{0*}(Y) = i : C_*(\tilde{Y}^1) \to C_*(X)$.

III.2.1. **Proposition.**
a) $E^1_{st} = H_t(\tilde{Y}^s)$ and d^1 is i_*, induced by i.
b) $E^r_{st} \Longrightarrow H_{s+t}(X, Y, \mathbb{Z})$, inducing a weight filtration on $H_*(X, Y, \mathbb{Z})$. $E^\infty_{st} \otimes \mathbb{Q} = Gr^W_{-t} H_{s+t}(X, Y, \mathbb{Q})$ (in Deligne's weight notation).
c) $E^*_{**} \otimes \mathbb{Q}$ degenerates at level two, i.e. $d^r \otimes 1_\mathbb{Q} = 0$ for $r \geq 2$.

Proof. The proof is similar as in the case of III.1.2. In this case, $K^\cdot_\mathbb{Q}$ is

$$0 \to n_*\mathbb{Q}_{\tilde{Y}^0} \to n_*\mathbb{Q}_{\tilde{Y}^1} \to \cdots .$$

Actually this (and the whole cohomologically mixed Hodge complex of (X,Y)) can be constructed as a mixed cone of the complexes of X, respectively of Y. This is compatible with the construction of $A_{**}(X,Y)$. □

III.3. **The homological double complex of** $(X, X-Y)$ **or** $(U,\partial U)$**.**

Here U is the "tubular neighbourhood" of Y. We define

$$A_{s,t}(X, X-Y) := C^{\pitchfork}_{t+2(s-1)}(\tilde{Y}^{-(s-1)}),$$

with $s \leq 0$ and $t+2(s-1) \geq 0$. Then ∂ and $\cap$ act as $\partial : A_{s,t} \to A_{s,t-1}$ and $\cap : A_{s,t} \to A_{s-1,t}$, hence $D = \partial + \cap$ is the differential of the total complex $\mathrm{Tot}_*(A_{**}(U,\partial U))$. Corollary II.3.6 reads as:

III.3.1. **Corollary.** $\Pi_\star^{-1} : (Tot_*(A_{**}(X, X-Y)), D) \to (C_{*-1}(\partial Z), \partial)$ *is a morphism of complexes i.e.* $\partial \Pi_\star^{-1} = \Pi_\star^{-1} D$.

Moreover, II.2.1, II.2.5 and II.2.8 and Poincaré duality imply the following result.

III.3.2. **Proposition.**

a) $E^1_{s+1,t} = H_{t+2s}(\tilde{Y}^{-s})$*, and* d_1 *is the transfer map* $i_!$.

b) $E^r_{st} \Longrightarrow H_{s+t}(X, X-Y, \mathbb{Z})$*, and* $E^\infty_{st} \otimes \mathbb{Q} = Gr^W_{-t} H_{s+t}(X, X-Y, \mathbb{Q})$.

c) $E^*_{**} \otimes \mathbb{Q}$ *degenerates at level two, i.e.* $d^r \otimes 1_{\mathbb{Q}} = 0$ *for* $r \geq 2$.

d) The Poincaré duality map pd (cf. II.2.6) induces an isomorphism of spectral sequences (for any $r \geq 1$*) between the cohomological spectral sequnence of* Y *and the above homological spectral sequence of* $(X, X-Y)$*, which provides exactly the ismorphism* $\cap[X] : Gr^W_{2n-t} H^{2n-s-t}(Y) \to Gr^W_{-t} H_{s+t}(X, X-Y)$.

e) $\Pi_\star^{-1}$ *from III.3.1 induces the boundary operator* $H_*(X, X-Y, \mathbb{Z}) = H_*(U, \partial U, \mathbb{Z}) \to H_{*-1}(\partial U, \mathbb{Z})$.

Proof. The main result here is the degeneration of the spectral sequence at level 2, which follows via duality *d*) from the *NCD* case.

Fix a triangulation T and let T' be its first barycentric subdivision. Then the Poincaré duality map (cf. II.2.6) can be organized in the following morphism of double complexes.

Let $A^T_{st}(Y) = C^T_t(\tilde{Y}^{s+1})$, which form a double complex with ∂ and i, similarly as in III.1. Set the double complex $B^T_{st}(Y) = Hom(A^T_{st}(Y), \mathbb{Z})$ with dual morphisms δ and i^*. Let $A^{T'}_{s+1,t}(X, X-Y) = C^{\pitchfork, T'}_{t+2s}(\tilde{Y}^{-s})$ with boundary morphisms ∂ and $\cap$ (similarly as $A_{**}(X, X-Y)$ defined above). Then $pd : B^T_{-s,2n-t}(Y) \to A^{T'}_{st}(X, X-Y)$ satisfies $\partial \circ pd = pd \circ \delta$ (cf. II.2.8) and $\cap \circ pd = pd \circ i^*$ (cf. II.2.9).

Now, $A_{**}(Y)$ is quasi-isomorphic to $A^T_{**}(Y)$ (i.e. their spectral sequences are the same for $r \geq 1$), and the later is dual to $B^T_{**}(Y)$. Using II.2.9, *pd* induces an isomorphism at the level of the E^1 term, hence it is an quasi-isomorphism, and it induces isomorphism at the level of any E^r ($r \geq 2$). On the other hand, $E^r(A^{T'}_{**}(X, X-Y)) = E^r(A_{**}(X, X-Y))$ for $r \geq 1$. Hence the result follows. □

III.4. **The homological double complex of $X - Y$.**

Define $A_{s,t}(X-Y) := C^{\pitchfork}_{t+2s}(\tilde{Y}^{-s})$, with $s \leq 0$ and $t + 2s \geq 0$. Then $D = \partial + \cap$ is a differential of the total complex $\mathrm{Tot}_*(A_{**}(X-Y))$.

Similarly to the case of the pair (X,Y), we can define $A^{\pitchfork}_{**}(X)$ by $A^{\pitchfork}_{s*}(X) = C^{\pitchfork}_{*}(X)$ if $s = 0$ and $= 0$ otherwise. Then $A_{s*}(X-Y) = A^{\pitchfork}_{s*}(X) \oplus A_{s+1,*}(X, X-Y)$. Actually, $A_{*-1,*}(X-Y)$ is the cone of the morphism $\cap : A^{\pitchfork}_{**}(X) \to A_{**}(X, X-Y)$, where $\cap$ is the intersection for $s = 0$ and zero otherwise.

Hence the Poincaré duality in III.3.2 extends to:

III.4.1. **Proposition.**
a) $E^1_{s,t} = H_{t+2s}(\tilde{Y}^{-s})$, *and* d_1 *is the transfer map* $i_!$.
b) $E^r_{st} \Longrightarrow H_{s+t}(X-Y, \mathbb{Z})$, *and* $E^{\infty}_{st} \otimes \mathbb{Q} = Gr^W_{-t} H_{s+t}(X-Y, \mathbb{Q})$.
c) $E^*_{**} \otimes \mathbb{Q}$ *degenerates at level two, i.e.* $d^r \otimes 1_{\mathbb{Q}} = 0$ *for* $r \geq 2$.
d) The Poincaré duality map pd (cf. II.4.6) induces an isomorphism of spectral sequences (for any $r \geq 1$*) between the cohomological spectral sequnence of* (X,Y) *and the above homological spectral sequence of* $X - Y$*, which provides exactly the ismorphism* $\cap[X] : Gr^W_{2n-t} H^{2n-s-t}(X,Y) \to Gr^W_{-t} H_{s+t}(X-Y)$.

III.4.2. **Remark.** Notice that by corollary II.3.6 and III.4.1

$$\Pi^{-1}_{\star} : (Tot_*(A_{**}(X-Y)), D) \to (C_*(Z), \partial)$$

is a quasi-isomorphism of complexes.

Notice also that $H_*(C_*(Z)) = H_*(X-Y)$. Indeed, if Z_α is a small collar of Z, then $H_*(Z) = H_*(Z - Z_\alpha) = H_*(Z - \partial Z) = H_*(X-Y)$.

The above morphism can be extended to the pair $(X, X-Y)$ as follows. Consider now the complex (Ker_{*+1}, ∂) defined as the kernel of the morphism $\Pi_* : (C_*(\partial Z), \partial) \to (C_*(Y), \partial)$ induced by Π. In fact, this complex coincide with the kernel of the morphism $\Pi_* : (C_*(Z), \partial) \to (C_*(X), \partial)$ since Π induces an isomorphism over $X - Y$.

III.4.3. **Corollary.**
a) $\Pi^{-1}_{\star} : A_{s,t}(X-Y) \to C_{s+t}(Z)$ *maps* $A_{s+1,t}(X, X-Y) \subset A_{s,t}(X-Y)$ *in* Ker_{s+t+1}.
b) There is a natural commutative diagram whose vertical arrows are quasi-isomorphisms.

$$\begin{array}{ccccccccc}
0 & \to & Tot_*(A_{*+1,*}(X, X-Y)) & \to & Tot_*(A_{**}(X-Y)) & \to & Tot_*(A^{\pitchfork}_{**}(X)) & \to & 0 \\
 & & \downarrow \Pi^{-1}_{\star} & & \downarrow \Pi^{-1}_{\star} & & \downarrow j_1 & & \\
0 & \to & Ker_{*+1} & \to & C_*(Z) & \to & C_*(X) & \to & 0
\end{array}$$

The horizontal lines induce the long homology exact sequence of the pair $(X, X-Y)$.

Proof. The image of any cycle $\xi \in C^{\pitchfork}_k(\tilde{Y}^p)$, with $p \geq 1$, via the composite map

$$C^{\pitchfork}_k(\tilde{Y}^p) \xrightarrow{\Pi_p^{-1}} C_{k+p}(\partial Z) \xrightarrow{\Pi_*} C_{k+p}(Y)$$

has k-dimensional support. Actually, this support is the image of $|\xi|$ by the natural map $\tilde{Y}^p \to Y$. Hence it supports no cycle in $C_{k+p}(Y)$. $\square$

IV. Review of the spectral sequence associated with a double complex

IV.1. Let $A = \oplus_{s,t} A_{s,t}$ be a finite homological double complex with operators

$$\begin{tikzcd} A_{s-1,t} & A_{s,t} \arrow[l, "\delta"'] \arrow[d, "\partial"] \\ & A_{s,t-1} \end{tikzcd} \qquad \delta^2 = 0,\ \partial^2 = 0,\ \delta\,\partial + \partial\,\delta = 0$$

Let $\mathrm{Tot}_k\, A = \oplus_{s+t=k} A_{s,t}$ and $D = d + \delta$ be the associated total complex. Define the (weight) filtration by $W_\ell A = \oplus_{s \le \ell} A_{s,t}$. By general theory, there is a spectral sequence $(E^r_{s,t}, d_r)_{r\ge 1}$, converging to the graded homology of A. We put $E^0_{s,t} = A_{s,t}$; then

$$E^1_{s,t} = \mathrm{Ker}(\partial : A_{s,t} \to A_{s,t-1})/Im(\partial : A_{s,t+1} \to A_{s,t})$$

and d^1 is induced at the homology level by δ. Let

$$\begin{aligned} Z^r_{s,t} &= \{c \in W_s \,\mathrm{Tot}_{s+t}\, A : Dc \in W_{s-r} A\} \\ Z^r_s &= \{c \in W_s A : Dc \in W_{s-r} A\} \\ Z^\infty_{s,t} &= \{c \in W_s \,\mathrm{Tot}_{s+t}\, A : Dc = 0\} \\ Z^\infty_s &= \{c \in W_s A : Dc = 0\}. \end{aligned}$$

Then

$$\begin{aligned} E^r_{s,t} &= Z^r_{s,t}/(Z^{r-1}_{s-1} + DZ^{r-1}_{s+r-1}) \\ E^\infty_{s,t} &= Z^\infty_{s,t}/(Z^\infty_{s-1} + DA \cap W_s A). \end{aligned}$$

In particular, $E^r_{s,t}$ is generated by the class of elements

$$c_{st} + c_{s-1,t+1} + \cdots + c_{s-r+1,t+r-1} \in Z^r_{st} \quad (c_{pq} \in A_{pq})$$

which satisfy

$$(*_r) \qquad \partial c_{st} = 0; \quad \partial c_{s-1,t+1} + \delta c_{st} = 0; \quad \ldots ; \quad \partial c_{s-r+1,t+r-1} + \delta c_{s-r+2,t+r-2} = 0.$$

The map d^r can be defined as:

$$d^r_{s,t}[c_{st} + c_{s-1,t+1} + \cdots + c_{s-r+1,t+r-1}] = [\delta c_{s-r+1,t+r-1}]$$

For $r = 1$, this means that $c_{st} \in A_{pq}$ with $\partial c_{st} = 0$. For $r = 2$, Z^2_{st} is generated by elements of type $c_{st} + c_{s-1,t+1}$ such that $\partial c_{s,t} = 0, \partial c_{s-1,t+1} + \delta c_{s,t} = 0$, which generates $ker d^1$ on E^1. For $r = \infty$, E^∞_{st} is generated by class of cycles of type

$$c_{st} + c_{s-1,t+1} + \cdots + c_{s-r,t+r} + \cdots$$

satisfying

$$(*_\infty) \qquad \partial c_{st} = 0; \quad \partial c_{s-r,t+r} + \delta c_{s-r+1,t+r-1} = 0 \text{ for any } r \ge 1.$$

IV.1.1. Now, assume that $d^2 = d^3 = \cdots = 0$ which means $E^2_{st} \equiv E^\infty_{st}$. Then consider

$$\mathrm{Ker}(d_1 : E^1_{st} \to E^1_{s-1,t}) \to E^\infty_{st}.$$

This shows that for any $c^2 = c_{st} + c_{s-1,t+1}$ satisfying $(*_2)$, there is a class $c^\infty = c'_{st} + c'_{s-1,t+1} + \cdots$ with $(*_\infty)$, such that $[c^2] = [c^\infty]$ in E^∞_{st}. More precisely, we show that we can choose the cycle c^∞ with $c'_{st} = c_{st}$.

IV.1.2. **Proposition.**

a) *Consider a class $[c_{st}] \in \mathrm{Ker}[d_1 : E^1_{st} \to E^1_{s-1,t}]$ and fix a representative $c_{st} \in A_{st}$ with $\partial c_{st} = 0$ and $\delta c_{st} \in \partial A_{s-1,t+1}$ i.e., we can define a cycle $c^2 = c_{st} + c_{s-1,t+1} \in Z^2_{st}$. If $d^2 = 0$, then c^2 can be replaced by $\tilde{c}_2 = c_{st} + \tilde{c}_{s-1,t+1}$ such that $[c^2] = [\tilde{c}^2]$ in E^2_{st} and $\tilde{c}^2$ can be completed to a cycle $\tilde{c}^3 = c_{st} + \tilde{c}_{s-1,t+1} + \tilde{c}_{s-2,t+2} \in Z^3_{st}$.*

The fact that $d^2 = 0$ is equivalent to the fact that this can be done for any cycle c^2.

b) *More generally, consider an element $c^r = \sum_{i=0}^{r-1} c_{s-i,t+i} \in Z^r_{s,t}$ and assume that $d^r = 0$. Then c^r can be replaced by $\tilde{c}^r = c_{st} + \sum_{i=1}^{r-1} \tilde{c}_{s-i,t+i} \in Z^r_{s,t}$ with $[c^r] = [\tilde{c}^r] \in E^r_{s,t}$ such that $\tilde{c}^r$ can be completed to $\tilde{c}^{r+1} = \tilde{c}^r + \tilde{c}_{s-r,t+r} \in Z^{r+1}_{st}$.*

Again, in the case of an arbitrary spectral sequence, if any c_r can be completed to $\tilde{c}^{r+1}$ as above, then $d^r = 0$.

c) *In particular, if $d^2 = d^3 = \cdots = 0$, then*

   i) *any representative c_{st} (with $\partial c_{st} = 0$ and $\delta c_{st} \in \partial A_{s-1,t+1}$) of a class in $\mathrm{Ker}(d_1 | E^1_{st})$ can be completed to a cycle*

$$c^\infty_{s,t} = c_{s,t} + \sum_{i \geq 1} c_{s-i,t-i} \in Z^\infty_{st};$$

   ii) *if c^∞_{st} and ${c'}^\infty_{st}$ are two liftings of c_{st} as in (i) , then $c^\infty_{st} - {c'}^\infty_{st} \in Z^\infty_{s-1}$;*

   iii) *if c_{st} and c'_{st} are representatives of the same class of $\mathrm{Ker}(d_1 | E^1_{st})$ (i.e. $c_{st} - c'_{st} \in \partial B_{s,t+1}$) and we complete c_{st} and c'_{st} to c^∞_{st} and ${c'}^\infty_{st}$, respectively, as in (i), then*

$$c^\infty_{st} - {c'}^\infty_{st} \in DA \cap W_s + Z^\infty_{s-1}.$$

V. Constructing rational cycles

The aim of the present section is to use the additional degeneration property of the weight spectral sequences and the above review in order to give a construction of cycles which represent $W_{-t}H_{s+t}(A, B)$, where $(A, B) = (Y, \varnothing)$, (X, Y), $(X, X - Y)$, $(X - Y, \varnothing)$ (The case $(\partial U, \varnothing)$ is again separated in section VI.). Moreover, we find topological properties and characterizations of these cycles. We will keep the notations of the previous sections. All the homology groups are considered with rational coefficients.

V.1. **Cycles in** $(X, X - Y)$**.**

V.1.1. First notice that $H_*(X, X-Y) = H_*(U, \partial U)$. In this subsection we will construct relative cycles in $(U, \partial U)$, i.e chains ξ supported in U with $|\partial\xi| \subset \partial U$.

For any (s,t) with $s+t=k$ fix $c_{st} \in A_{st}(X, X-Y)$ with $\partial c_{st} = 0$ and $\cap c_{st} \in im\partial$. Then it can be completed to a cycle $c^\infty_{s,t} = c_{st} + c_{s-1,t+1} + \cdots$ with $Dc^\infty_{s,t} = 0$. Consider $\Pi^{-1}_\star c^\infty_{s,t} \in C_{k-1}(\partial Z)$. Then by III.3.1 one has $\partial\Pi^{-1}_\star c^\infty_{s,t} = 0$. Fix a sub-analytic homeomorphism $\phi_\alpha : \partial Z \times [0,\alpha] \to Z_\alpha$, where Z_α is a collar of $\partial Z \subset Z$ (cf. I), and write $U = \Pi\circ\phi_\alpha(\partial Z \times [0,\alpha])$. For any chain $\xi \in C_*(\partial Z)$, one can associate in a natural way a new chain $\xi \times [0,\alpha] \in C_{*+1}(\partial Z \times [0,\alpha])$. Then $\Pi_* \circ (\phi_\alpha)_*(\Pi^{-1}_\star c^\infty_{s,t} \times [0,\alpha])) \in C_k(U)$ has boundary supported in $\partial U = \Pi \circ \phi_\alpha(\partial Z \times \{\alpha\})$. For simplicity, we will denote this relatice cycle by $c^{rel}_{s,t}$.

Notice that for $p \geq 1$, $|c^{rel}_{s,t}| \cap Y^p = |c_{s',k-s'}|$, where $s' := \min\{s, -p+1\}$. More precisely, if $p \geq 1-s$, the intersection $|c^{rel}_{s,t}| \cap Y^p$ is $|c_{-p+1,k+p-1}|$, which has dimension $k-p-1$. For $p = -s$, the intersection $|c^{rel}_{s,t}| \cap Y^p$ is $|c_{s,t}|$, which has dimension $k-p-2$.

V.1.2. **Proposition.**

a) Fix $k = s+t$. To any homology class $[c_{st}]$ in $H_{t+2(s-1)}(\tilde{Y}^{-s+1})$ with $[c_{st}] \cap \tilde{Y}^{-s+2} = 0$, the above construction gives a relative cycle $c^{rel}_{s,t}$. Their classes generate $W_{-t}H_k(X, X-Y) = W_{-t}H_k(U, \partial U)$ (and they are well-defined modulo $W_{-t-1}H_k$).

b) $W_{-t}H_k(X, X-Y) = \{[c]:$ c a relative cycle in X with $|\partial c| \subset X-Y$ such that $\dim |c| \cap Y^p = k-p-1$ for $p \geq 1-k+t$, and $< k-p-1$ for $p = -k+t\}$. In these intersection conditions we always assume that $p \geq 1$, otherwise the assumption is considered empty. In particular, $W_{-k}H_k(X, X-Y) = H_k(X, X-Y)$.

The above formula is also equivalent to the following characterization:

$$W_{-t}H_k(X, X-Y) = \{[c] : \dim |c| \cap Y^{-k+t} < 2k-t-1.\}$$

This shows that the homeomorphism type of the pair (X, Y) (or even of $(U, \partial U)$) determines completely the weight filtration of $H_*(X, X-Y)$.

V.1.3. **The support filtration.** Let $U^p \subset X$ be a small regular neighbourhood of Y^p in X (cf. [R-S], ch. 3). Then for U sufficiently small (with respect to U^p), and $p \geq 1$, one can consider the group

$$S_pH_k(U, \partial U) := im\left(H_k(U^p, U^p \cap \partial U) \to H_k(U, \partial U)\right).$$

It is not difficult to see that the (decreasing) filtration $\{S_pH_k(U, \partial U)\}_p$ is independent on the choice of neighbourhoods U^p; it will be called the “support filtration” of $H_k(U, \partial U)$.

In the above construction, $|c^{rel}_{s,t}| \subset U^p$ (where $p = 1-s$ and $s+t=k$), therefore:

$$W_{-k-p+1}H_k(U, \partial U) \subset S_pH_k(U, \partial U).$$

In general, the inclusion is strict.

V.2. **Cycles in $X - Y$.**

V.2.1. Here we will use the identification $H_*(X-Y)=H_*(Z)$. Similarly as above, for any (s,t) with $s+t=k$ fix $c_{st}\in A_{st}(X-Y)$ with $\partial c_{st}=0$ and $\cap c_{st}\in im\partial$. Then it can be completed to a cycle $c^\infty_{s,t}=c_{st}+c_{s-1,t+1}+\cdots$ with $Dc^\infty_{s,t}=0$. Consider $\Pi_\star^{-1}c^\infty_{s,t}\in C_k(Z)$ (cf. III.4.2).

V.2.2. **Proposition.**
a) The cycles $\Pi_\star^{-1}c^\infty_{s,t}$ *generate* $W_{-t}H_k(Z)=W_{-t}H_k(X-Y)$ *(and they are well-defined modulo* W_{-t-1}*).*
b) $W_{-k}H_k(X-Y)=H_k(X-Y)$*. For* $t<-k$*,* $W_{-t}H_k(X-Y)$ *can be characterized by the exact sequence of the pair* $(X,X-Y)$*:*

$$H_{k+1}(X)\to H_{k+1}(X,X-Y)\to H_k(X-Y)\to H_k(X).$$

Above $H_i(X)$ *is pure of weight* $-i$*, and the weight filtration of* $H_{k+1}(X,X-Y)$ *is characterized in the previous subsection.*

For a more detailed and geometrical presentation of the cycles in $X-Y$, see theorem III.0.

The above result shows that the homeomorphism type of the pair (X,Y) determines completely the weight filtration of $H_*(X-Y)$. On the other hand, it is well known that one cannot recover the weight filtration from the homeomorphism type (even the analytic type) of $X-Y$. For a counterexample, see e.g. [S-S], (2.12).

V.2.3. **The support filtration.** Similarly as above, let U^p be a regular neighbourhood of Y^p in X for any $p\geq 1$, and for $p=0$ take $U^0=X$. Define for any $p\geq 0$:

$$S_pH_k(X-Y):=im\,\big(H_k(U^p\cap(X-Y))\to H_k(X-Y)\big).$$

This provides the decreasing "support filtration" $S_*H_k(X-Y)$. Then:

$$W_{-k-p}H_k(X-Y)\subset S_pH_k(X-Y),$$

and the inclusion, in general, is strict.

V.3. **Cycles in Y.**

V.3.1. Consider $(A_{**}(Y),\partial,i)$ and fix a pair (s,t) with $s+t=k$. If one wants to construct a closed cycle in Y, it is natural to start with a chain $c_{st}\in C_t(\tilde{Y}^{s+1})$ (with $\partial c_{st}=0$) and tries to extent it. The first obstruction is $i(c_{st})\in im\partial$, i.e. the existence of $c_{s-1,t+1}$ with $i(c_{st})+\partial c_{s-1,t+1}=0$. Then the second obstruction is $i(c_{s-1,t+1})\in im\partial$, and so one. The remarkable fact is that once the first obstruction is satisfied then all the others are automatically satisfied. This follows from the degeneration of the spectral sequence (III.1.2), and it is a consequence of the algebraicity of Y; in a simple topological context it is not true.

If $c_{st}\in A_{st}(Y)=E^0_{st}$ satisfies $\partial c_{st}=0$ and $i(c_{st})\in im\partial$, then it can be completed to a cycle

$$c^\infty_{st}=c_{st}+c_{s-1,t+1}+\cdots+c_{0,k}$$

with $Dc^{\infty}_{st} = 0$ (see section IV). Let $n : \tilde{Y}^p \to Y$ be again the natural projection for any $p \geq 1$. Then the correspondence $c_{st} \mapsto n_*(c_{0,k}) \in C_*(Y)$ defines a closed cycle in Y. (We invite the reader to verify that $\partial n_*(c_{0,k}) = 0$.)

V.3.2. **Proposition.** *Fix $k = s + t$. To any homology class $[c_{st}] \in H_t(\tilde{Y}^{s+1})$ with $i[c_{st}] = 0$ one can construct a cycle*

$$c^{\infty}_{st} = c_{st} + c_{s-1,t+1} + \cdots + c_{0,k}$$

with $Dc^{\infty}_{st} = 0$. The homology classes $[n_*c_{0,k}] \in H_k(Y)$ generate $W_{-t}H_k(Y)$. Any other completion c^{∞}_{st} of c_{st} provides the same homology class modulo $W_{-t-1}H_k(Y)$.

Actually, the chains $c_{s-i,t+i}$ can be recovered from the intersections $|c_{0,k}| \cap Y^{s-i+1}$. Indeed, for any i with $s > i \geq 0$, and write $p = s - i + 1$:

$$|n_*c_{0,k}| \cap Y^p = n(|c_{p-1,k-p+1}|), \text{ whose dimension is } k - p + 1.$$

On the other hand, if $p \geq s + 2$, then in the homology class $[c_{st}]$ one can take a representative c_{st} which is in a general position with respect to the Y^p (i.e. they are transversal), therefore:

$$|n_*c_{0,k}| \cap Y^p = |n_*c_{st}| \cap Y^p \text{ whose dimension is } t - 2(p - s - 1) < k - p + 1.$$

V.3.3. **Corollary.** *The weight filtration of $H_k(Y)$ can be characterized by:*

$$W_{-t}H_k(Y) = \{[c] : \dim |c| \cap Y^p = k - p + 1 \textit{ for } \ p \leq k - t + 1,$$

$$\textit{and } < k - p + 1 \textit{ for } \ p = k - t + 2.\}$$

Actually, this is equivalent to the following characterization:

$$W_{-t}H_k(Y) = \{[c] : \dim |c| \cap Y^{k-t+2} < t - 1.\}$$

This can be rewritten in the language of intersection homology as follows. For any integer $s \geq 0$, consider the perversity $\underline{s}$ defined by $\underline{s}(2i) = i$ for $0 \leq i \leq s$, and $\underline{s}(2i) = s$ for $i \geq s$. (Since we have no stratum with odd codimension, $\underline{s}(2i + 1)$ is unimportant.) Notice that $\underline{s}$ is not a perversity in the sense of [GM] (i.e. does not satisfy $\underline{s}(2) = 0$), it is a *generalized perversity* (see. e.g. [K] or [H-S]).

V.3.4. **Corollary.** $W_{-k+s}H_k(Y) = im\,(IH^{\underline{s}}_k(Y) \to H_k(Y))$.

The above corollary in fact says that Deligne's weight filtration and the Zeeman filtration (or "support filtration") coincide. This fact was conjectured by MacPherson, and verified by C. McCrory in [McC] (see also [GNPP] and [H-S]). One of the main consequences of this fact is that the weight filtration of Y is completely topological.

In the last part of this subsection, we present a different construction for the case Y.

V.3.5. **Milnor's construction.** Sometimes it is convenient to replace Y by another space, called the "geometric realization" of the "semi-simplicial object" $\{\tilde{Y}^p\}_p$. The construction is done by Milnor [M] (see also [S], [D] and [A]).

This new space is homotopic to Y. It can be constructed as follows. Consider the natural (normalization) map $n : \tilde{Y}^1 \to Y$. For any points $y \in Y$ with $y \in Y_{\alpha_1,\dots,\alpha_j}$ and $y \notin Y_\beta$ for $\beta \notin \{\alpha_1, \cdots, \alpha_j\}$, the set $n^{-1}(y)$ consists of j points $\{y_k\}_k$. Obviously, if for any y, we identify in $\tilde{Y}^1$ the points $\{y_k\}_k$, then the quotient-space is exactly Y. But we want a slightly different construction: for any y, we glue to $\tilde{Y}^1$ a $(j-1)$–simplex $\Delta_y = [y_1, \dots, y_j]$, in a compatible way. The new space SY will have the following properties. The map $sn : SN \to Y$, defined by $sn(\Delta_y) = y$ is continuous, it extends n (i.e. $sn|\tilde{Y}^1 = n$), and it is a homotopy equivalence.

Actually, SY can be written as

$$SY = \coprod_{p\geq 1} \tilde{Y}^p \times \Delta_{p-1} \ / \sim,$$

where Δ_{p-1} is the $(p-1)$–simplex, and $\sim$ is a suitable identification. If for any $0 \leq j \leq p-1$, $i_j : \tilde{Y}^p \to \tilde{Y}^{p-1}$ denote the natural maps induced by the inclusions $Y_{\alpha_1,\dots,\alpha_p} \hookrightarrow Y_{\alpha_1,\dots,\hat{\alpha}_{j+1},\cdots,\alpha_p}$, and $\partial_j : \Delta_{p-2} \to \Delta_{p-1}$ are the face maps, then we identify:

$$(i_j y, x) \sim (y, \partial_j x), \text{ for any } 0 \leq j \leq p-1, \ y \in \tilde{Y}^p, \text{ and } \ x \in \Delta_{p-2}.$$

For more details, see [M], [G] or [A] (with even some pictures on page 239).

One of the advantages of the space SY is the existence of a morphism $sn^{-1} : C_k(\tilde{Y}^p) \to C_{k+p-1}(SY)$. This is defined as follows. For any k–simplex σ we will denote by the same symbol σ the chain with support σ and coefficient one. We will define $sn^{-1}(\sigma)$ by the chain

$$[\sigma \times \Delta_{p-1}] \subset \coprod \tilde{Y}^p \times \Delta_{p-1} \ / \sim .$$

(Here $[\sigma \times \Delta_{p-1}]$ can be covered by simplices, and we take all of them with coefficient one; this is the wanted chain.) This generates a linear map sn^{-1}.

V.3.6. **Lemma.** *The above map induces a morphism of complexes:*

$$ns^{-1} : (Tot_*(A_{**}(Y)), D) \to (C_*(SY), \partial)$$

which is a quasi-isomorphism.

Proof. For the first part notice that

$$\partial(\sigma \times \Delta_{p-1}) = \partial\sigma \times \Delta_{p-1} + (-1)^k \sigma \times \sum_i (-1)^i \partial_i \Delta_{p-1} = \partial\sigma \times \Delta_{p-1} + i(\sigma) \times \Delta_{p-2},$$

i.e. $\partial(sn^{-1}(\sigma) = sn^{-1}(\partial\sigma) + sn^{-1}(i(\sigma))$ (cf. the identification $\sim$ above). The second part follows from the above lemma III.1.2 (and its proof). □

Then III.1.2 and IV.1.2 read as:

V.3.7. **Proposition.**

a) If $c_{st} \in A_{st}(Y) = E^0_{st}$ satisfies $\partial c_{st} = 0$ and $i(c_{st}) \in im\partial$, then it can be completed to a cycle

$$c^\infty_{st} = c_{st} + c_{s-1,t+1} + \cdots + c_{0,k}$$

with $Dc^\infty_{st} = 0$. This means (cf. III.1.3) that $\partial ns^{-1}c^\infty_{st} = 0$, generating a homology class in $H_k(SY) = H_k(Y)$.

b) The homology classes $[ns^{-1}c^\infty_{st}]$ generate $W_{-t}H_k(Y)$. Any other completion c^∞_{st} of c_{st} provides the same homology class modulo $W_{-t-1}H_k(Y)$.

The cycle $ns^{-1}(c^\infty_{st})$ constructed above can be projected via sn into Y. In this case, all the chains $c_{s-i,t+i}$ (for $s > i \geq 0$) provide "degenerate" chains in Y (with support of dimension $t+i$). This shows that the correspondence $c_{st} \mapsto sn^{-1}(c^\infty_{st}) \in C_*(SY)$ can be replaced by $c_{st} \mapsto n_*(c_{0,k}) \in C_*(Y)$.

V.4. **Cycles in (X, Y).**

V.4.1. The construction is similar to the previous case. Take $c_{st} \in A_{st}(X, Y)$ with $\partial c_{st} = 0$ and $i(c_{st}) \in im\partial$. Then complete to $c^\infty_{st} = c_{st} + \cdots + \tilde{c}_{0,k}$, where $k = s + t$. Here $\tilde{c}_{0,k} \in C_k(X)$ with $\partial\tilde{c}_{0,k} = -i(c_{1,k-1})$, hence $|\partial\tilde{c}_{0,k}| \subset Y$ provided that $s > 0$.

V.4.2. **Proposition.**

a) The relative cycles $\tilde{c}_{0,k}$ (associated with c_{st} as above) generate $W_{-t}H_k(X, Y)$, and are well-defined modulo $W_{-t-1}H_k(X, Y)$.

b) $W_{-t}H_k(X, Y) = \{[c] :$ c a relative cycle in X with $|\partial c| \subset Y$ such that $\dim |c| \cap Y^p = k - p$ for $p \leq k - t$, and $< k - p$ for $p = k - t + 1\}$. This is equivalent to the following characterization:

$$W_{-t}H_k(X, Y) = \{[c] : \dim |c| \cap Y^{k-t+1} < t - 1\}.$$

c) If $\partial : H_k(X, Y) \to H_{k-1}(Y)$ is the natural map, then $W_{-t}H_k(X, Y) = \partial^{-1}W_{-t}H_{k-1}(Y)$. In particular, $W_*H_*(X, Y)$ is topological. Moreover, it can also be characterized by

$$W_{-t}H_k(X, Y) = \{[c] : \dim |\partial c| \cap Y^{k-t+1} < t - 1\}.$$

V.4.3. Let i' be the composite of the natural maps $C_*(SY) \to C_*(Y) \to C_*(X)$ (induced by sn and the inclusion). Then there is the following commutative diagram of complexes:

$$\begin{array}{ccccccccc}
0 & \to & Tot_*(A_{**}(X)) & \to & Tot_*(A_{**}(X, Y)) & \to & Tot_*(A_{*-1,*}(Y)) & \to & 0 \\
 & & \Big\downarrow 1 & & \Big\downarrow 1 \oplus sn^{-1} & & \Big\downarrow sn^{-1} & & \\
0 & \to & C_*(X) & \to & Cone_*(i') & \to & C_{*-1}(SY) & \to & 0
\end{array}$$

It is clear that the second line provides the long homology exact sequence of the pair (X, Y), and the vertical arrows in the diagram are quasi-isomorphisms.

V.5. **The homology of Y^p and $(X, X - Y^p)$.**

V.5.1. We end this section with the discussion of some connections between the weight filtration and the homology of the spaces Y^p and $(X, X-Y^p)$. The proofs are standard or similar to the proofs already presented, therefore we will omit them. If A_{**} is a double complex, we denote by $\sigma_{s\leq i}A_{**}$ the subcomplex of A_{**} defined by $(\sigma_{s\leq i}A_{**})_{st} = A_{st}$ for $s\leq i$ and zero otherwise. $\sigma_{s\geq i+1}A_{**}$ is the quotient double complex $A_{**}/\sigma_{s\leq i}A_{**}$.

V.5.2. We start with the double complex of Y. For any $p\geq 1$:

$$H_k(Tot_*(\sigma_{s\geq p-1}\,A_{**}(Y))) = H_{k-p+1}(Y^p).$$

Moreover, $W_*(\sigma_{s\geq p-1}A_{**}(Y))$ induces the weight filtration on this space. The exact sequence:

$$0\to \sigma_{s\leq p-2}A_{**}(Y)\to A_{**}(Y)\to \sigma_{s\geq p-1}A_{**}(Y)\to 0$$

provides the identity:

$$W_{p-2-k}H_k(Y) = \text{ Ker }\big(\,H_k(Y)\xrightarrow{b} H_{k-p+1}(Y^p)\,\big),$$

where b is the ("Mayer-Vietoris") boundary map (associated with the closed covering $Y=\cup_i Y_i$).

Similar identities are valid for $A_{**}(X,Y)$, in particular, for $p\geq 1$:

$$W_{p-1-k}H_k(X,Y) = \text{ Ker }\big(\,H_k(X,Y)\xrightarrow{b'} H_{k-p}(Y^p)\,\big),$$

where b' is the composite of the boundary operator and b.

In the case of $(X,X-Y)$, for $p\geq 1$, one has:

$$H_k(Tot_*(\sigma_{s\leq -p+1}\,A_{**}(X,X-Y))) = H_{k+p-1}(X,X-Y^p),$$

and $W_*(\sigma_{s\leq -p+1}A_{**}(X,X-Y))$ induces the weight filtration on this space. The exact sequence:

$$0\to \sigma_{s\leq -p+1}A_{**}(X,X-Y)\to A_{**}(X,X-Y)\to \sigma_{s\geq -p+2}A_{**}(X,X-Y)\to 0$$

provides:

$$W_{-p+1-k}H_k(X,X-Y) = \,im\,\big(\,H_{k+p-1}(X,X-Y^p)\xrightarrow{b} H_k(X,X-Y)\,\big),$$

where b again is a ("Mayer-Vietoris") boundary map (associated with the open covering $(X,X-Y_i)_i$).

Similar property holds for $A_{**}(X-Y)$, in particular, for $p\geq 1$:

$$W_{-p-k}H_k(X-Y) = \,im\,\big(\,H_{k+p}(X,X-Y^p)\xrightarrow{b'} H_k(X-Y)\,\big),$$

where b' is the composite of b and the boundary map.

The above properties of the pairs $(Y,\varnothing)$ and $(X,X-Y)$, respectively of (X,Y) and $(X-Y,\varnothing)$ correspond by the Poincaré Duality. This follows also from the commutativity of some diagrams like the following:

$$\begin{array}{ccc} H^{2n-k}(Y) & \xrightarrow{PD} & H_k(X, X-Y) \\ \Big\uparrow b^* & & \Big\uparrow b \\ H^{2n-k-p+1}(Y^p) & \xrightarrow{PD} & H_{k+p-1}(X, X-Y^p) \end{array}$$

Above, PD denotes the Poincaré Duality $\cap[X]$ (induced by pd).

VI. The homology of ∂U.

VI.1. **The homological double complex of ∂U.**

VI.1.1. The space ∂U appears in a natural way in two homological exact sequences. One of them is the pair $(U, \partial U)$. The homological exact sequence of this pair, and the isomorphism $H_*(U) = H_*(Y)$ suggests that a possible double complex for ∂U should satisfy

$$0 \to A_{*+1,*}(U, \partial U) \to A_{**}(\partial U) \to A_{**}(Y) \to 0.$$

Here, the double complexes of $(U, \partial U)$, by definition is the same as the double complex of $(X, X-Y)$. Now, using this and the double complex of Y, one can try to define the double complex of ∂U by

$$A_{s,t}(\partial U) = \begin{cases} C^{\pitchfork}_{t+2s}(\tilde{Y}^{-s}) & \text{for } s \le -1 \\ C_t(\tilde{Y}^{s+1}) & \text{for } s \ge 0. \end{cases}$$

But now we are confronted with the definition of the arrows of the double complex. Actually, we have all the vertical arrows, and all the horizontal arrows corresponding to $s \le -1$ and $s \ge 0$. But we need to define a map $C_*(\tilde{Y}^1) \to C^{\pitchfork}_{*-2}(\tilde{Y}^1)$ with some nice properties. First of all, this map should be compatible with the other arrows, in the sense that the whole complex should form a double complex. On the other hand, we expect (since we know the cohomological E_1 term of the corresponding spectral sequence) that this map should induce at the homology level the "intersection matrix"; more precisely, the map $\oplus_\alpha c_\alpha \mapsto \oplus_\alpha d_\alpha$, where for $c_\alpha \in H_k(Y_\alpha)$ one gets $d_\alpha = \sum_\beta c_\beta \cap [Y_\alpha]$. Here, if $\alpha \neq \beta$, then $c_\beta \cap [Y_\alpha]$ is provided by the transfer map $i_! : H_k(Y_\beta) \to H_{k-2}(Y_{\alpha,\beta})$ composed with $i_* : H_{k-2}(Y_{\alpha,\beta}) \to H_{k-2}(Y_\alpha)$ induced by the inclusion. If $\alpha = \beta$, then $c_\alpha \cap [Y_\alpha]$ is the cap product by $[Y_\alpha]$.

Therefore, the wanted map $C_*(\tilde{Y}^1) \to C^{\pitchfork}_{*-2}(\tilde{Y}^1)$ should be some kind of intersection, but we realize immediatelly that we face serious obstructions: we have to intersect cycles which are not "transversal", and the image should be special "transversal" cycle. Even if we try to modify our complexes, similar type of obstruction will survive. The explanation for this is the following. The cap product $c_\alpha \cap [Y_\alpha]$ cannot be determined only from the spaces $\{\tilde{Y}^p\}_{p\ge 1}$, we need the Poincaré dual of the spaces Y_α in X, hence we need also to consider the space X (or at least U) in our double complex.

So, we have to think about ∂U as the boundary of Z, and we have to consider the pair of spaces $(Z, \partial Z)$. Notice that $H_*(Z, \partial Z) = X_*(X, Y)$ and $H_*(Z) = H_*(X-Y)$, so it is natural to ask for for a double complex with:

$$0 \to A_{*+1,*}(X, Y) \to A_{**}(\partial U) \to A_{**}(X-Y) \to 0.$$

The construction is done in the next subsection.

VI.1.2. **The construction of $A_{**}(\partial U)$.** Consider the following diagram of complexes:

$$
\begin{array}{ccccccccc}
\cdots & C^{\pitchfork}_{*-4}(\tilde{Y}^2) & \xleftarrow{\cap} & C^{\pitchfork}_{*-2}(\tilde{Y}^1) & \xleftarrow{\cap} & C^{\pitchfork}_{*}(\tilde{Y}^0) & & & \\
 & & & & & \Big\downarrow j_1 & & & \\
 & & & & & C_*(\tilde{Y}^0) & \xleftarrow{i} & C_*(\tilde{Y}^1) & \xleftarrow{i} & C_*(\tilde{Y}^2) & \cdots
\end{array}
$$

The first line corresponds to the double complex $A_{**}(X-Y)$ (where the column $s=0$ is $C^{\pitchfork}_{*}(\tilde{Y}^0)=C^{\pitchfork}_{*}(X)$), and the second line is the double complex $A_{**}(X,Y)$. Notice that j_1 can be considered as a morphism of double complexes, hence the usual cone construction provides the double complex $A_{**}(\partial U)=\{A_{s*}(\partial U)\}_s$:

$$
\begin{array}{ccccccccc}
\cdots & C^{\pitchfork}_{*-4}(\tilde{Y}^2) & \xleftarrow{\cap} & C^{\pitchfork}_{*-2}(\tilde{Y}^1) & \xleftarrow{\cap} & C^{\pitchfork}_{*}(\tilde{Y}^0) & & & \\
 & & & \oplus & \swarrow^{j_1} & \oplus & & & \\
 & & & C_*(\tilde{Y}^0) & \xleftarrow{i} & C_*(\tilde{Y}^1) & \xleftarrow{i} & C_*(\tilde{Y}^2) & \cdots \\
 & & & & & & & & \\
 & s=-2 & & s=-1 & & s=0 & & s=1 &
\end{array}
$$

VI.1.3. **Proposition.**

a) The E^1 term of the spectral sequence associated with $(A_{**}(\partial U),W)$ is:

$$
\begin{array}{ccccccccc}
\cdots & H_{*-4}(\tilde{Y}^2) & \xleftarrow{\cap} & H_{*-2}(\tilde{Y}^1) & \xleftarrow{\cap} & H_{*}(\tilde{Y}^0) & & & \\
 & & & \oplus & \swarrow^{id} & \oplus & & & \\
 & & & H_*(\tilde{Y}^0) & \xleftarrow{i_*} & H_*(\tilde{Y}^1) & \xleftarrow{i_*} & H_*(\tilde{Y}^2) & \cdots \\
 & & & & & & & & \\
 & s=-2 & & s=-1 & & s=0 & & s=1 &
\end{array}
$$

where $\cap$ is given in II.2.5, e.g. $d\in H_k(X)$:

$$\cap(d)=\oplus_\beta\,(-1)^k d\cap[Y_\beta]\in\oplus_\beta H_{k-2}(Y_\beta)=H_{k-2}(\tilde{Y}^1);$$

and i_* is induced by i (cf. III.1), e.g. for $\oplus_\alpha c_\alpha\in\oplus_\alpha H_k(Y_\alpha)=H_k(\tilde{Y}^1)$:

$$i_*(\oplus_\alpha c_\alpha)=\sum_\alpha(-1)^k\,i_\alpha(c_\alpha)\,\in H_k(X).$$

The above E^1 term is quasi-isomorphic to the complex:

$$\cdots H^{\pitchfork}_{*-4}(\tilde{Y}^2)\xleftarrow{\cap}H^{\pitchfork}_{*-2}(\tilde{Y}^1)\xleftarrow{I}H_*(\tilde{Y}^1)\xleftarrow{i_*}H_*(\tilde{Y}^2)\cdots,$$

where $I:=\cap\circ i_*$, i.e.

$$I(\oplus_\alpha c_\alpha)=\oplus_\beta\,(\sum_\alpha\,i_\alpha(c_\alpha)\cap[Y_\beta]\,).$$

b) $E^r_{st}\Longrightarrow H_{s+t}(\partial U,\mathbb{Z})$ and $E^\infty_{st}\otimes\mathbb{Q}=Gr^W_{-t}H_{s+t}(\partial U,\mathbb{Q})$.
c) $E^*_{**}\otimes\mathbb{Q}$ degenerates at level two, i.e. $d^r\otimes 1_{\mathbb{Q}}=0$ for $r\geq 2$.

Proof. For *a*) and *b*) use II.2.1 and II.2.5 and the corresponding definitions. *c*) follows from the results of the previous subsections and from the construction of the mixed cone. □

Now, we would like to construct a morphism of complexes $Tot_*(A_{**}(\partial U)) \to C_*(\partial U)$. For this the above complex is not convenient, because of the presence of the global terms $C_*^{\pitchfork}(\tilde{Y}^0)$ and $C_*(\tilde{Y}^0)$. In the next construction, we replace these complexes by the complexes of chains supported by the close neighbourhood U. More precisely, we define $A'_{**}(\partial U)$:

$$\begin{array}{ccccccccc}
\cdots & C^{\pitchfork}_{*-4}(\tilde{Y}^2) & \overset{\cap}{\leftarrow} & C^{\pitchfork}_{*-2}(\tilde{Y}^1) & \overset{\cap}{\leftarrow} & C^{\pitchfork}_{*}(U) & & & \\
 & & & \oplus & \overset{j_1}{\swarrow} & \oplus & & & \\
 & & & C_*(U) & \overset{i}{\leftarrow} & C_*(\tilde{Y}^1) & \overset{i}{\leftarrow} & C_*(\tilde{Y}^2) & \cdots
\end{array}$$

Then $E^1(A'_{**})$ is:

$$\begin{array}{ccccccccc}
\cdots & H_{*-4}(\tilde{Y}^2) & \overset{\cap}{\leftarrow} & H_{*-2}(\tilde{Y}^1) & \overset{\cap}{\leftarrow} & H_{*}(U) & & & \\
 & & & \oplus & \overset{id}{\swarrow} & \oplus & & & \\
 & & & H_*(U) & \overset{i_*}{\leftarrow} & H_*(\tilde{Y}^1) & \overset{i_*}{\leftarrow} & H_*(\tilde{Y}^2) & \cdots
\end{array}$$

which is quasi-isomorphic to $E^1(A_{**})$, hence $E^r(A_{**}) = E^r(A'_{**})$ for any $r \geq 2$.

VI.1.4. Now, the advantage of this second double complex lies in the following construction. Let $A_{**}(Z_\alpha)$ be the double complex:

$$\begin{array}{cccccccccc}
\cdots & C^{\pitchfork}_{*-4}(\tilde{Y}^2) & \overset{\cap}{\leftarrow} & C^{\pitchfork}_{*-2}(\tilde{Y}^1) & \overset{\cap}{\leftarrow} & C^{\pitchfork}_{*}(U) & \leftarrow & 0 & \cdots \\
 & s=-2 & & s=-1 & & s=0 & & s=1 &
\end{array}$$

Then, there is a natural morphisms of complexes $\Pi_\star^{-1} : Tot_*(A_{**}(Z_\alpha)) \to C_*(Z_\alpha)$ (cf. II.3.8). This can be inserted in the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \to & Tot_*(A_{*+1,*}(X, X-Y)) & \to & Tot_*(A_{**}(Z_\alpha)) & \to & C^{\pitchfork}_*(U) & \to & 0 \\
 & & \big\downarrow \Pi_\star^{-1} & & \big\downarrow \Pi_\star^{-1} & & \big\downarrow j_1 & & \\
0 & \to & Ker_{*+1} & \to & C_*(Z_\alpha) & \to & C_*(U) & \to & 0
\end{array}$$

Since, using a sub-analytic homeomorphism $Z_\alpha \approx \partial U \times [0, \alpha]$, the second line in the above diagram induces the long homology exact sequence of the pair $(U, \partial U)$.

VI.1.5. **Corollary.** *The vertical arrows in the above diagram are quasi-isomorphisms.*

VI.2. Cycles in ∂U.

VI.2.1. The complex $A_{**}(\partial U)$ was important from theoretical point of view: its spectral sequence is degenerating at rank two. On the other hand, the complex $A'_{**}(\partial U)$ has the same E^r term for $r \geq 2$, hence the degeneration property is still valid, but has the advantage that it is local: X is replaced by the neighbourhood U of Y. In the above construction we will use this second double complex A'_{**}.

Notice that $\sigma_{s\leq -1}A_{**}(X-Y) = A_{*+1,*}(X, X-Y)$ is a subcomplex of $A'_{**}(\partial U)$. The corresponding pair provides the exact sequence:

VI.2.2. $$\to H_{k+1}(X, X-Y) \xrightarrow{\partial} H_k(\partial U) \xrightarrow{\alpha} H_k(Y) \xrightarrow{\delta} H_k(X, X-Y) \to \ .$$

Since the weights of $H_{k+1}(X, X-Y)$ are $\leq -k-1$, and the weights of $H_k(Y)$ are $\geq -k$, the weight filtration of $H_k(\partial U)$ is completely determined from the above exact sequence and from the weight filtrations of $(X, X-Y)$ and Y. For $l \leq -k-1$: $W_l H_k(\partial U) = \partial(W_l H_{k+1}(X, X-Y)$, and for $l \geq -k$: $W_l H_k(\partial U) = \alpha^{-1} W_l H_k(Y)$.

Notice again that the weight filtration of $H_k(\partial U)$ is completely determined from the topology of the pair (X, Y). On the other hand, it is impossible to determine it from the diffeomorphism type of ∂U. For counterexamples, see again [S-S], e.g. (3.1). (For this one has to use the fact that the link of on isolated singularity (S, s) can be identified with the boundary ∂U , where $U \to S$ is a resolution of the singular point and S is a Stein representative of (S, s).)

Corresponding to the above discussion of the weights, in the construction of the cycles in ∂U we will distinguish two different cases as well.

Fix a pair (s, t) with $s + t = k$ and $s \leq -1$. Consider $c_{st} \in A_{s+1,t}(X, X-Y) \subset A_{s,t}(\partial U)$ with $\partial c_{st} = 0$ and $\cap c_{st} \in im\partial$. Complete to $c^\infty_{s,t}$ with $Dc^\infty_{s,t} = 0$. Then (cf. III.3.1) $\Pi^{-1}_\star c^\infty_{s,t} \in C_k(\partial Z)$ is closed. Recall that we have a natural identification of ∂Z and ∂U.

VI.2.3. **Proposition.** *For $s \leq -1$, the cycles $\Pi^{-1}_\star c^\infty_{s,t}$ generate $W_{-t}H_k(\partial U)$.*

Moreover, $[\Pi^{-1}_\star c^\infty_{s,t}] = \partial[c^{rel}_{s,t}]$, where $c^{rel}_{s,t}$ is constructed in V.1. In particular, these homology classes inherit all the properties of $W_* H_{k+1}(X, X-Y)$ (including their relationship with the "support" filtration). The details are left to the reader.

Now, assume that $s \geq 0$. In this case the construction of the cycles is more involved: we have to lift some cycles from Y to ∂U.

Assume that $c'_{st} \in A'_{st}(\partial U)$ ($s \geq 0$) satisfies $\partial c'_{st} = 0$ and $d_1[c'_{st}] = 0$. If $s = 0$ this means that $c'_{st} = c'_{0k} = c^\perp_{0,k} + c_{0,k}$, where $c^\perp_{0,k} \in C^\pitchfork_k(U)$ and $c_{0,k} \in C_k(\tilde{Y}^1)$ such that $\partial c^\perp_{0,k} = \partial c_{0,k} = 0$ and $i_*(c_{0,k}) + c^\perp_{0,k} + \partial\gamma = 0$ for some $\gamma \in C_{k+1}(U)$.

If $s > 0$, then $c'_{st} \in A'_{st}(\partial U)$ has "only one component": $c'_{st} = c_{st} \in C_t(\tilde{Y}^{s+1})$ with $\partial c_{st} = 0$ and $i_* c_{st} \in im\partial$.

In both cases c'_{st} can be completed to a cycle

$$c^\infty_{s,t} = c_{s,t} + c_{s-1,t+1} + \cdots + c_{1,k-1} + (c_{0,k} + c^\perp_{0,k}) + (\gamma_{-1,k+1} + c_{-1,k+1}) + c_{-2,k+2} + \cdots$$

where $c_{s,t} \in C_t(\tilde{Y}^{s+1})$ for $s \geq 0$, $c_{s,t} \in C^\pitchfork_{t+2s}(\tilde{Y}^{-s})$ for $s \leq -1$, $c^\perp_{0,k} \in C^\pitchfork_k(U)$, and $\gamma_{-1,k+1} \in C_{k+1}(U)$. These chains satisfies the following relations:

$\partial c_{s,t} = 0$;
$i_* c_{s-l,t+l} + \partial c_{s-l-1,t+l+1} = 0$ for $0 \leq l \leq s-1$;
$\partial c^{\perp}_{0,k} = 0$;
$i_* c_{0,k} + j c^{\perp}_{0,k} + \partial \gamma_{-1,k+1} = 0$;
$\cap c^{\perp}_{0,k} + \partial c_{-1,k+1} = 0$;
$\cap c_{l,k-l} + \partial c_{l-1,k-l+1} = 0$ for $l \leq -1$.

Now, we will separate the chain (for the notations, see VI.1.4):

$$\bar{c}_{st} := c^{\perp}_{0,k} + c_{-1,k+1} + c_{-2,k+2} + \cdots \in A_{**}(Z_\alpha).$$

Then by the above relations, $D(\bar{c}_{st}) = 0$, where here D is the differential in $A_{**}(Z_\alpha)$. Therefore, $\Pi^{-1}_{\star}\bar{c}_{st} \in C_k(Z_\alpha)$ is closed. Obviously, the projection $pr : Z_\alpha \approx \partial Z \times [0,\alpha] \to \partial Z$ provides a closed cycle $pr_*\Pi^{-1}_{\star}\bar{c}_{st} \in C_k(\partial Z)$.

VI.2.4. **Proposition.** *For any $s \geq 0$ and $t+s=k$, the cycles $pr_*\Pi^{-1}_{\star}\bar{c}_{st}$ generate $W_{-t}H_k(\partial U)$.*

Notice that in the cycles $\bar{c}_{st}$ we do not see the chains $c_{l,k-l}$ for $l \geq 0$, in particular neither $c_{s,t}$, the chain which generates $\bar{c}_{st}$. The chain c_{st} is completed to a sequence $c_{st} + c_{s-1,t+1} + \cdots + c_{0,k}$ with $c_{0,k} \in C_k(\tilde{Y}^1)$. The chain $i_* c_{0,k}$ actually is closed (in U) and supported by Y. Now, this is replaced by the transversal chain $c^{\perp}_{0,k} \in C^{\pitchfork}_k(U)$ such that $i_* c_{0,k} + c^{\perp}_{0,k} + \partial\gamma = 0$ for some γ, and finally $c^{\perp}_{0,k}$ is completed to $\bar{c}_{st}$. It is really remarkable that the above algebraic construction plays the role of a very geometric operation: it replaces a closed chain supported in Y by another closed chain supported in U, homologous with the original one in U, and dimensionally transversal to the stratification given by Y (i.e. it solves the problem, obstruction mentioned in VI.1.1.)

VI.2.5. **Example.** Assume that $n=2$, and Y is a connected set of curves $\{Y_i\}_i$ in X. Set $g = \sum_i \mathrm{genus}(Y_i)$. If Γ is the dual graph of the curves then let $c_\Gamma = rank H_1(|\Gamma|)$ be the number of independent cycles in $|\Gamma|$. Let I be the intersection matrix of the irreducible curves Y_i. Then it is well-known that $rank H_1(\partial U) = rank\,\mathrm{Ker}\, I + 2g + c_\Gamma$. These three contributions correspond exactly to the weight of $H_1(\partial U)$.

Indeed, for $s=-1$, take $c_{-1,2} \in C^{\pitchfork}_0(\tilde{Y}^1)$ as above. A possible chain $c_{-1,2}$ is an arbitrary point P in $Y^1 - Y^2$ with coefficient one. Then $c^{\infty}_{-1,2} = c_{-1,2}$ and $\Pi^{-1}(c_{-1,2})$ is a circle S^1, the loop around Y in a transversal slice at P. These loops γ_P generate $W_{-2}H_1$ (isomorphic to *coker* I). If $s=0$, consider a closed 1-cycle $c_{0,1}$ in one of the components of Y. This can be changed by a homologous cycle $c^{\perp}_{0,1}$ in U, which has no intersection with Y. Hence it can be contracted to ∂U. These cycles generate $W_{-1}H_1$ (so that $\dim Gr^W_{-1}H_1 = 2g$). Notice that the lifting $c_{0,1} \mapsto c^{\perp}_{0,1}$ is defined modulo the cycles of type γ_P. Finally, consider $c_{1,0} \in C_0(Y^2)$ such that $d_1[c_{1,0}] = 0$. Here $d_1 : H_0(Y^2) \to H_0(\tilde{Y}^1)$ and $\mathrm{Ker}\, d_1 \approx H_1(|\Gamma|)$. Take $c_{0,1} \in C_1(\tilde{Y}^1)$ such that $ic_{1,0} + \partial c_{0,1} = 0$. Then $i(c_{0,1})$ is a cycle in U supported by Y. We replace it by $c^{\perp}_{0,1}$ which has no intersection with Y. They provide the remaining cycles in W_0H_1 so that $\dim Gr^W_0 H_1 = c_\Gamma$.

Now, we discuss the case $H_2(\partial U)$ as well. Take a point $P \in Y^2$ (with coefficient one) corresponding to $c_{-2,4}$. Then $\Pi^{-1}c_{-2,4}$ is a torus in ∂U. They generate $W_{-4}H_2$. Notice that $W_{-4}H_2 \approx coker \cap : H_2(\tilde{Y}^1) \to H_0(Y^2)$ has dimension c_Γ. Next, take a generic closed 1-cycle in $\tilde{Y}^1$ whose image c in Y has no intersection with Y^2. The 2-cycle $\Pi^{-1}(c)$ in ∂U is an S^1 bundle over c; they generate $W_{-3}H_2$. Finally, consider $c_{02} = \sum_i m_i Y_i \in C_2(\tilde{Y}^1)$ such that $\cap[c_{02}] = 0$. This means that $[c_{02}] \cdot [Y_j] = 0$ for all j. The dimension of the space generated by these classes $[c_{02}]$ is *coker* I. Now, each c_{02} is replaced by a transversal 2-cycle $c_{02}^{\perp}$. Transversality implies that $c_{02}^{\perp} \cap Y_i$ is a 0-cycle in Y_i, which by the above assumption is zero-homologous. In particular, $c_{02}^{\perp} \cap Y_i = \partial c_{-1,3}$. Now, take a very small tubular neighbourhood U' of $Y^1 - Y^2$ with projection $pr : U' \to Y^1 - Y^2$. (Here $U - U'$ stays for Z_α.) Then the boundaries of the chains $c_{02}^{\perp} - U'$ and the S^1-bundle $(pr|\partial U')^{-1}(c_{-1,3})$ can be identified (modulo sign) so they can be glued. They generate the remaining classes in $W_{-2}H_2$.

VI.2.6. **Remark. – Purity results.** The above example shows that in general all the possible weights (permitted by the spectral sequence) can appear. For example, if ∂U is the boundary (or link) of a 1-parameter family of projective curves over a small disc, then the intersection matrix I has 1-dimensional kernel, hence $H_1(\partial U)$ can have weight $-2, -1$ and 0. On the other hand, if ∂U is the boundary of the resolution of a normal surface singularity, then I is non-degenerate, hence the non-trivial weight of $H_1(\partial U)$ are -1 and 0.

More generally, if S is a projective algebraic variety with unique singular point $s \in S$, and $\phi : X \to S$ is a resolution of this isolated singularity with normal crossing exceptional divisor $Y = \phi^{-1}(s)$, then the following additional restrictions hold. They can be easily deduced from the corresponding cohomological statements about the weight filtrations of the links of isolated singularities. For a complete proof and details see e.g. [N-A].

1) If $k \le n-1$ then $Gr_l^W H_k(\partial U) = 0$ for l not in $[-k, 0]$;
2) If $k \ge n$ then $Gr_l^W H_k(\partial U) = 0$ for l not in $[-2n, -k-1]$;
3) If $k \ge n$ then $H_k(Y)$ is pure of weight $-k$ and $H_k(U) \to H_k(U, \partial U)$ is injective;
4) If $k \le n$ then $H_k(U, \partial U)$ is pure of weight $-k$ and $H_k(U) \to H_k(U, \partial U)$ is surjective;
5) Consider the exact sequence (cf. the E^1 term in VI.1.3):

$$\cdots H^{\pitchfork}_{k-4}(\tilde{Y}^2) \overset{\cap}{\leftarrow} H^{\pitchfork}_{k-2}(\tilde{Y}^1) \overset{I_k}{\leftarrow} H_k(\tilde{Y}^1) \overset{i_*}{\leftarrow} H_k(\tilde{Y}^2) \cdots .$$

If $k \ge n$ then $\operatorname{Ker} I_k = im\, i_*$, if $k \le n$ then $\operatorname{Ker} \cap = im\, I_k$.

Actually, between the above exact sequence and the the exact sequence VI.2.2 there is the following connection: δ is non-trivial only for weight $-k$ and $Gr^W_{-k}\delta$ can be identified with $\hat{I}_k : H_k(\tilde{Y}^1)/im\, i_* \to \operatorname{Ker} \cap$.

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